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Integrable spin chain in superconformal Chern-Simons theory

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Abstract: $\mathcal{N} = 6$ superconformal Chern-Simons theory was proposed as gauge theory dual to Type IIA string theory on $\mathrm{AdS}_4 \otimes \mathbb{CP}^3$. We study integrability of the theory from conformal dimension spectrum of single trace operators at planar limit. At strong 't Hooft coupling, the spectrum is obtained from excitation energy of free superstring on $\mathrm{OSp}(6|4;\mathbb{R})/\,\mathrm{SO}(3,1) \times \mathrm{SU}(3) \times \mathrm{U}(1)$ supercoset. We recall that the worldsheet theory is integrable classically by utilizing well-known results concerning sigma model on symmetric space. With R-symmetry group SU(4), we also solve relevant Yang-Baxter equation for a spin chain system associated with the single trace operators. From the solution, we construct alternating spin chain Hamiltonian involving three-site interactions between $\mathbf{4}$ and $\overline{\mathbf{4}}$. At weak 't Hooft coupling, we study gauge theory perturbatively, and calculate action of dilatation operator to single trace operators up to two loops. To ensure consistency, we computed all relevant Feynman diagrams contributing to the dilatation opeator. We find that resulting spin chain Hamiltonian matches with the Hamiltonian derived from Yang-Baxter equation. We further study new issues arising from the shortest gauge invariant operators $\mathrm{Tr}Y^I Y^\dagger_J = (\mathbf{15}, \mathbf{1})$. We observe that 'wrapping interactions' are present, compute the true spectrum and find that the spectrum agrees with prediction from supersymmetry. We also find that scaling dimension computed naively from alternating spin chain Hamiltonian coincides with the true spectrum. We solve Bethe ansatz equations for small number of excitations, and find indications of correlation between excitations of $\mathbf{4}$'s and $\overline{\mathbf{4}}$'s and of nonexistence of mesonic $(\mathbf{4}\overline{\mathbf{4}})$ bound-state.

Keywords: Gauge-gravity correspondence, Chern-Simons Theories.

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Contents

1. Introduction

In a recent remarkable development, Aharony, Bergman, Jafferis and Maldacena (ABJM) [1] made a new addition to the list of microscopic AdS/CFT correspondence [2]: three-dimensional $\mathcal{N} = 6$ superconformal Chern-Simons theory dual to Type IIA string theory on $\mathrm{AdS}_4 \times \mathbb{CP}^3$ [3]. Both sides of the correspondence are characterized by two integer-valued coupling parameters N and k. On the superconformal Chern-Simons theory side, they are the rank of product gauge group $\mathrm{U}(N) \times \overline{\mathrm{U}(N)}$ and Chern-Simons levels $+k, -k$, respectively. On the Type IIA string theory side, they are related to spacetime curvature and dilaton gradient or Ramond-Ramond flux, all measured in string unit. Much the same way as the counterpart between $\mathcal{N} = 4$ super Yang-Mills theory and Type IIB

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string theory on $\mathrm{AdS}_5 \times \mathbb{S}^5$, we can put the new correspondence into precision tests in the planar limit:

$$N \to \infty, \qquad k \to \infty \qquad \text{with} \qquad \lambda \equiv \frac{N}{k} \quad \text{fixed} \tag{1.1}$$

by interpolating 't Hooft coupling parameter λ between superconformal Chern-Simons theory regime at $\lambda \ll 1$ and semiclassical $\mathrm{AdS}_4 \times \mathbb{CP}^3$ string theory regime at $\lambda \gg 1$.

In the correspondence between $\mathcal{N} = 4$ super Yang-Mills theory and Type IIB string theory on $\mathrm{AdS}_5 \times \mathbb{S}^5$, the integrability structure first discovered by Minahan and Zarembo [4] led to remarkable progress in diverse fronts of the correspondence.[1] It is therefore interesting to examine if the new correspondence shows also an integrability structure. The purpose of this work is to demonstrate integrability structure inherent to the $\mathcal{N} = 6$ superconformal Chern-Simons theory of ABJM.[2]

AdS/CFT correspondence asserts that gauge invariant, single trace operators in superconformal Chern-Simons theory are dual to free string excitation modes in $\mathrm{AdS}_4 \times \mathbb{CP}^3$, valid at weak and strong 't Hooft coupling regime, respectively. In particular, conformal dimension of the operators should match with excitation energy of the string modes. The $\mathcal{N} = 6$ superconformal Chern-Simons theory has $\mathrm{SO}(6) \simeq \mathrm{SU}(4)$ R-symmetry and contains two sets of bi-fundamental scalar fields $Y^I, Y^\dagger_I$ $(I = 1, 2, 3, 4)$ that transform as $\mathbf{4}, \overline{\mathbf{4}}$ under SU(4). Therefore, the single trace operators take the form:

$$\begin{aligned} \mathcal{O} &= \mathrm{Tr}(Y^{I_1} Y^\dagger_{J_1} \cdots Y^{I_L} Y^\dagger_{J_L}) C^{J_1 \cdots J_L}_{I_1 \cdots I_L} \\ &= \overline{\mathrm{Tr}}(Y^\dagger_{J_1} Y^{I_1} \cdots Y^\dagger_{J_L} Y^{I_L}) C^{J_1 \cdots J_L}_{I_1 \cdots I_L} \ . \end{aligned} \tag{1.2}$$

In superconformal Chern-Simons theory, chiral primary operators, corresponding to the choice of (1.2) with $C^{J_1 \cdots J_L}_{I_1 \cdots I_L}$ totally symmetric in both sets of indices and traceless, form the lightest states. In free string theory on $\mathrm{AdS}_4 \times \mathbb{CP}^3$. Kaluza-Klein supergravity modes form the lightest states. In this work, we study conformal dimension of single trace operators and identify integrability structure organizing the excitation spectrum above the chiral primary or the Kaluza-Klein states.[3]

In section 2, we begin with recapitulating the standard argument for integrability of free string on $\mathrm{AdS}_4 \times \mathbb{CP}^3$ at $\lambda \to \infty$. Recalling the construction of [24] and utilizing the idea of [25], we argue that sigma model on $\mathrm{OSp}(6|4;\mathbb{R})/[\mathrm{SO}(3,1) \times \mathrm{SU}(3) \times \mathrm{U}(1)]$ supercoset has commuting monodromy matrices and infinitely many conserved nonlocal charges. In section 3, we begin main part of this work. Guided by earlier development in $\mathcal{N} = 4$ super Yang-Mills counterpart, we assume integrability and solve Yang-Baxter equations for R-matrices between $\mathbf{4}$ and $\overline{\mathbf{4}}$ sites in (1.2). From corresponding transfer matrices, we then find the Hamiltonian takes the form of one-parameter family of 'alternating spin chain', whose variants were studied previously in different contexts [26]–[30]. In section

[1] Selected but nonexhaustive list of contributions in this subject include [5] - [21]. For a comprehensive mid-development review, see [13].

[2] Integrability in $\mathcal{N} \leq 3$ superconformal Chern-Simons theory was investigated by Gaiotto and Yin [22] previously. Their tentative result indicated otherwise.

[3] While bulk of this work was completed, we received parallel work in [23].

4, we study superconformal Chern-Simons theory of ABJM at $\lambda \to 0$ in perturbation theory. Pure Chern-Simons theory is free from any ultraviolet divergences since the theory is diffeomorphism invariant and hence topological . Once matter is coupled, as in ABJM theory, topological feature is lost and the quantum theory will receive nontrivial radiative corrections. As such, the single trace operators (1.2) will acquire nontrivial anomalous dimensions in general. In three dimensions, logarithmic ultraviolet divergence arises only at even loop orders. Therefore, the first nontrivial correction starts at two loops. We compute two-loop operator mixing and anomalous dimension matrix of the single trace operators (1.2). In dimensional reduction method, we compute the complete set of relevant Feynman diagrams and find that the two-loop anomalous dimension matrix matches with the integrable 'alternating spin chain' Hamiltonian derived in section 3. In section 5, we study a new important feature of the superconformal Chern-Simons theory compared to $\mathcal{N} = 4$ super Yang-Mills theory. Since the anomalous dimensions begin to arise from two loops and next-to-nearest sites, the shortest single trace operators of $L = 1$ will be subject to 'wrapping interactions'. The 'alternating spin chain' Hamiltonian does not describe spectrum of $L = 1$ operators, so we compute all relevant 'wrapping interaction' diagrams and construct the correct Hamiltonian for $L = 1$. Curiously, we find that the correct spectrum coincides with the naive spectrum computed from the 'alternating spin chain' Hamiltonian at $L = 1$. In section 6, utilizing results previously obtained for general A_{n-1} Lie algebras [31, 27, 30], we explicitly write down eigenvalues of the transfer matrices and Bethe ansatz equations of the 'alternating spin chain' we derived in section 3. To gain understanding how the 'alternating spins' behave, we solve the equations for a few simple situations. We find an indication for real-space correlations between excitations on $\mathbf{4}$ spin sites and those on $\overline{\mathbf{4}}$ spin sites, and for non-existence of meson-like $(\mathbf{4}\overline{\mathbf{4}})$ bound-states. We discuss various implications of these findings for general excitations. In particular, we argue that general excitations are more complex than the pattern emerging from closed SU(2) sub-sectors discussed recently [32, 33].

2. Integrable string from worldsheet sigma model

In this section, we set out a motivation for searching for integrability in $\mathcal{N} = 6$ superconformal Chern-Simons theory. The $\lambda \to \infty$ dual of this theory is Type IIA string theory on $\mathrm{AdS}_4 \otimes \mathbb{CP}^3$. The background is a direct product of symmetric spaces, AdS_4 and $\mathbb{CP}^3$. It is well known that the $(1 + 1)$-dimensional sigma model on symmetric space is classically integrable. So, at least for bosonic modes, we expect worldsheet dynamics of a free string on $\mathrm{AdS}_4 \otimes \mathbb{CP}^3$ is integrable at the classical level, $\lambda \to \infty$. In this section, we recapitulate this argument for the bosonic part and discuss how the construction to full superstring can be made.[4]

Bosonic part of string worldsheet Lagrangian on $\mathrm{AdS}_4 \otimes \mathbb{CP}^3$ is given by

$$I_{\rm b} = \frac{R^2}{4\pi} \int_\Sigma \sqrt{-h} h^{\alpha\beta} \Big[(D_\alpha X^m)^\dagger (D_\beta X^m) + (D_\alpha Z^a)^\dagger (D_\beta Z^a) \Big]. \tag{2.1}$$

[4]For other parallel works on classical integrability, see [34]–[36].

Here, we use embedding coordinates in $\mathbb{R}^{3,2}$ and $\mathbb{C}^4$ and describe $\mathrm{AdS}_4 = \mathrm{SO}(3,2)/\,\mathrm{SO}(3,1)$ and $\mathbb{CP}^3 = \mathrm{SU}(4)/\,\mathrm{SU}(3)\times\mathrm{U}(1)$ as G/H coset hypersurfaces:

$$\begin{aligned} &\mathrm{AdS}_4: \qquad (X^m) = (X^{-1}, X^1, X^2, X^3, X^0) \qquad \text{with} \qquad X^2 = 1\,, \\ &\mathbb{CP}^3: \qquad (Z^a) = (Z^1, Z^2, Z^3, Z^4)/\{\simeq, \mathbb{C}\} \qquad \text{with} \qquad |Z|^2 = 1, \end{aligned} \tag{2.2}$$

respectively. The hypersurface conditions are imposed by introducing auxiliary connection K_α, A_α and by defining covariant derivatives[5] $D_\alpha X^m \equiv \partial_\alpha X^m + iK_\alpha X^m$ and $D_\alpha Z^a \equiv \partial_\alpha Z^a + iA_\alpha Z^a$. These conditions imply that $X^m\partial_\alpha X_m = 0$ and $(D_\alpha Z^a)^\dagger Z^a = Z^{a\dagger}(D_\alpha Z^a) = 0$. Following [24], we first recapitulate basic aspects for classical integrability of sigma model on $\mathrm{AdS}_4 \times \mathbb{CP}^3$. Construction of the coset sigma model is facilitated by the coset elements:

$$G(\sigma) \equiv g(\sigma) \oplus \widetilde{g}(\sigma) = e^{i\pi P(\sigma)} \oplus e^{i\pi \widetilde{P}(\sigma)} \tag{2.3}$$

where $P(\sigma), \widetilde{P}(\sigma)$ are projection matrices onto respective one-dimensional subspaces. They are

$$\begin{aligned} P^{mn}(\sigma) &= X^m(\sigma)X^n(\sigma) \qquad \text{with} \qquad \delta_{mn}P^{mn}(\sigma) = 1, \\ \widetilde{P}^{ab}(\sigma) &= Z^{a\dagger}(\sigma)Z^b(\sigma) \qquad \text{with} \qquad \delta_{ab}P^{ab}(\sigma) = 1, \end{aligned} \tag{2.4}$$

respectively. By elementary algebra, we verify that

$$G(\sigma) = G^{-1}(\sigma) = (\mathbb{I}_5 - 2P(\sigma)) \oplus (\mathbb{I}_4 - 2\widetilde{P}(\sigma)). \tag{2.5}$$

Then, because $-8|D_\alpha X^m|^2 = \mathrm{Tr}(\partial_\alpha g \cdot \partial_\alpha g^{-1})$ for AdS_4 and $+4|D_\alpha Z^a|^2 = \mathrm{Tr}(\partial_\alpha \widetilde{g} \cdot \partial_\beta \widetilde{g}^{-1})$ for $\mathbb{CP}^3$, the worldsheet action (2.1) is expressible as

$$I_{\text{bosonic}} = \frac{R^2}{8}\int_\Sigma \sqrt{-h}h^{\alpha\beta}\Big[-\frac{1}{2}\mathrm{Tr}J_\alpha J_\beta + 2\mathrm{Tr}\widetilde{J}_\alpha\widetilde{J}_\beta\Big], \tag{2.6}$$

where $J = g^{-1}\mathrm{d}g$ and $\tilde{J} = \tilde{g}^{-1}\mathrm{d}\tilde{g}$, respectively. We shall choose the conformal gauge $\sqrt{-h}h^{\alpha\beta} = \delta^{\alpha\beta}$ on the worldsheet. This leads to Virasoro gauge condition

$$T_\pm \equiv -\frac{1}{4}(J_0 \pm J_1)^2 + (\tilde{J}_0 \pm \tilde{J}_1)^2 = 0\ . \tag{2.7}$$

The currents $J, \tilde{J}$ are conserved by equations of motion, and define tangent flows on the G/H coset space.

We now take group conjugation and transform the left-invariant currents $J, \widetilde{J}$ to the right-invariant currents: $(j, \widetilde{j}) = (g\cdot J\cdot g^{-1},\ \widetilde{g}\cdot\widetilde{J}\cdot\widetilde{g}^{-1})$. The equations of motion in conformal gauge are

$$\mathrm{d}^*j(\sigma) = 0 \qquad \text{and} \qquad \mathrm{d}^*\widetilde{j}(\sigma) = 0. \tag{2.8}$$

From the Bianchi identities, we also have

$$dj + j\wedge j = 0 \qquad \text{and} \qquad d\widetilde{j} + \widetilde{j}\wedge\widetilde{j} = 0. \tag{2.9}$$

[5] We introduced auxiliary connection K_α to treat AdS_4 in complete parallel to $\mathbb{CP}^3$.

Finally, Virasoro constraints are

$$-\frac{1}{4}(j_0 \pm j_1)^2 + (\tilde{j}_0 \pm \tilde{j}_1)^2 = 0. \tag{2.10}$$

We can solve these equations using the Lax representation. Consider the Lax derivative with flat connection $a(x)$ depending on a spectral parameter x:

$$D(x) = \mathrm{d} + a(x) \qquad \text{with} \qquad \mathrm{d}a + a \wedge a = 0. \tag{2.11}$$

Using (2.8), (2.9), we find that the most general form of the Lax connection is given by

$$a(x) = \frac{2}{x^2 - 1} j(\sigma) + \frac{2x}{x^2 - 1} {}^*j(\sigma) \qquad x \in \mathbb{C} + \{\infty\}/\{\pm 1\}. \tag{2.12}$$

and similarly construct $\widetilde{a}(x)$ from $\tilde{j}(\sigma)$. With the flat connection $A(x) \equiv (a(x), \widetilde{a}(x))$, consider the Wilson line

$$W[\gamma; x] = \mathcal{P}\exp\left(\int_\gamma A(x)\right) \tag{2.13}$$

As the connection $A(x)$ is flat, the eigenvalues of the Wilson line are independent of the choice of the contour γ. Thus, all Wilson lines commute each another and provides classical R-matrices obeying Yang-Baxter equations. Expanding in spectral parameter x, we then obtain infinitely many conserved nonlocal charges as moment of the power series:

$$\mathcal{Q}^{(n)} = \frac{1}{n!}\partial_x^n \int_\gamma \mathrm{d}\sigma A(x)\Big|_{x=0}. \tag{2.14}$$

This establishes that the sigma model on $\mathrm{AdS}_4 \times \mathbb{CP}^3$ is classically integrable.

We now discuss how the above consideration may be extended to Type IIA string on the supercoset:

$$\frac{\widehat{G}}{H} = \frac{\mathrm{OSp}(6|4;\mathbb{R})}{\mathrm{SO}(3,1) \times \mathrm{SU}(3) \times \mathrm{U}(1)}. \tag{2.15}$$

With the coefficients of current bilinears determined as in (2.6), we see immediately that the worldsheet Lagrangian is expressible as supertrace over the supergroup $\mathrm{OSp}(6|4;\mathbb{R})$:

$$I_{\text{supercoset}} = \frac{R^2}{8}\int_\Sigma \mathrm{Str}\,(\widehat{J} \wedge {}^*\widehat{J})\,. \tag{2.16}$$

Here, $\widehat{J}(\sigma) = \widehat{G}^{-1}(\sigma)\mathrm{d}\widehat{G}(\sigma)$ and $\widehat{G}(\sigma) = \exp(i\pi\widehat{P}(\sigma))$ is the supercoset element. This indicates that the bosonic action (2.6) is extendible straightforwardly to a supercoset action by adding 24 fermionic off-diagonal components to (2.3)–(2.5) and define super-projection matrix $\widehat{P}$ and supercoset element $\widehat{G}$ analogously.

Construction of infinitely many nonlocal currents requires a new condition to the supercoset. If the supergroup $\hat{G}$ permits $\mathbb{Z}_4$ grading under which the subgroup H is a fixed point set, the construction of [25] implies that a flat connection exists from which nonlocal

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currents can be constructed through the Lax formulation. From the embedding we constructed of, we have $\widehat{J} = J + Q$, where Q denotes fermionic current. For the supergroup we deal with, $\widehat{G} = \mathrm{OSp}(6|4;\mathbb{R})$, it is well known that $\widehat{G}$ admits no outer automorphism of order four [37]. However, one can easily construct a suitable $\mathbb{Z}_4$ inner automorphism. Since we need the subgroup H is a fixed point set, the automorphism can be defined as a product of two $\mathbb{Z}_2$ involutions on the defining representations of $\mathrm{SU}(4) \simeq \mathrm{SO}(6)$ and $\mathrm{Sp}(4)$ modulo overall reflection. This then ensures that the G/H current $\widehat{J} = Q_1 \oplus J \oplus Q_3$ is $\mathbb{Z}_4$ graded as $[1, 2, 3]$ and that infinitely many conserved nonlocal currents can be constructed accordingly.

At quantum level, the supergroup $\widehat{G}=$ $\mathrm{OSp}(6|4;\mathbb{R})$ has another nice feature that its Killing form vanishes identically. This means that sigma model on $\widehat{G}$ would be conformally invariant, at least, at one loop. We actually need to quotient $\widehat{G}$ by bosonic subgroup H and consider string worldsheet action on the supercoset $\widehat{G}/H$. This action in general breaks the conformal invariance. To restore the conformal invariance, a suitable Wess-Zumino term needs to be added. It was observed [38] that the requisite Wess-Zumino term can be constructed provided the bosonic subgroup H is a fixed point set of the $\mathbb{Z}_4$ grading of $\widehat{G}$. This is precisely the same condition that ensures the existence of a flat connection and infinitely many conserved charges thereof. Therefore, the supercoset sigma model is conformally invariant and can describe consistent string worldsheet dynamics, at least at one loop order in worldsheet perturbation theory.

Given such mounting evidences, it is highly likely that Type IIA string on $\mathrm{AdS}_4 \times \mathbb{CP}^3$ is integrable at $\lambda \to \infty$ and further extends to λ finite and even to weak coupling regime.[6] With such motivation, we now turn to the main part of this work and investigate integrability at the weak coupling regime, $\lambda \to 0$.

3. Integrable spin chain from Yang-Baxter

The $\mathrm{U}(N) \times \overline{\mathrm{U(N)}}$ invariant, single-trace operators under consideration

$$\begin{aligned} & \mathcal{O}^{(I)}{}_{(J)} \equiv \mathrm{Tr}(Y^{I_1} Y^\dagger_{J_1} \cdots Y^{I_L} Y^\dagger_{J_L}) \\ \simeq \ & \mathcal{O}_{(J)}{}^{(I)} \equiv \mathrm{Tr}(Y^\dagger_{J_1} Y^{I_1} \cdots Y^\dagger_{J_L} Y^{I_L}) \end{aligned} \tag{3.1}$$

are organized according to $\mathrm{SU}_R(4)$ irreducible representations. Operator mixing under renormalization and their evolution in perturbation theory can be described by a spin chain of total length $2L$. What kind of spin chain system do we expect? In this section, viewing the operators (3.1) as a spin chain system and utilizing quantum inverse scattering method, we shall derive spin chain Hamiltonian.

As is evident from the structure of operators (3.1), the prospective spin chain involves two types of $\mathrm{SU}_R(4)$ spins: $\mathbf{4}$ at odd lattice sites and $\overline{\mathbf{4}}$ at even lattice sites. It is thus natural to expect that the prospective spin chain is an 'alternating SU(4) spin chain' consisting of interlaced $\mathbf{4}$ and $\overline{\mathbf{4}}$. To identify the spin system and extract its Hamiltonian, it is imperative to solve inhomogeneous Yang-Baxter equations of $\mathrm{SU}_R(4)$ $\mathfrak{R}$-matrices with

[6] We note that the no-go theorem of Goldschmidt and Witten [39] for quantum conservation laws is evaded in the present case since the isotropy subgroup is not simple, and may lead to quantum anomalies [40].

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varying representations on each site. In fact, a general procedure for solving Yang-Baxter equations in this sort of situations is already set out in [26]. By construction, resulting spin chain system will be integrable. In this section, we shall follow this procedure and find that the putative SU(4) spin chain is an 'alternating spin chain' involving next-to-nearest neighbor interactions nested with nearest neighbor interactions.[7]

We first introduce $\mathfrak{R}^{\mathbf{44}}(u)$ and $\mathfrak{R}^{\mathbf{4\bar{4}}}(u)$, where the upper indices denote SU(4) representations of two spins involved in 'scattering process' and u, v denote spectral parameters. We demand these R-matrices to satisfy two sets of Yang-Baxter equations:

$$\mathfrak{R}^{\mathbf{44}}_{12}(u-v)\,\mathfrak{R}^{\mathbf{44}}_{13}(u)\,\mathfrak{R}^{\mathbf{44}}_{23}(v) = \mathfrak{R}^{\mathbf{44}}_{23}(v)\,R^{\mathbf{44}}_{13}(u)\,R^{\mathbf{44}}_{12}(u-v) \tag{3.2}$$
$$\mathfrak{R}^{\mathbf{44}}_{12}(u-v)\,\mathfrak{R}^{\mathbf{4\bar{4}}}_{13}(u)\,\mathfrak{R}^{\mathbf{4\bar{4}}}_{23}(v) = \mathfrak{R}^{\mathbf{4\bar{4}}}_{23}(v)\,\mathfrak{R}^{\mathbf{4\bar{4}}}_{13}(u)\,\mathfrak{R}^{\mathbf{44}}_{12}(u-v) \tag{3.3}$$

Here, the lower indices i, j denote that the $\mathfrak{R}$ matrix is acting on i-th and j-th site $V_i \otimes V_j$ of the full tensor product Hilbert space $V_1 \otimes V_2 \otimes \cdots \otimes V_{2L}$. We easily find that the R-matrices solving (3.2), (3.3) are given by

$$\mathfrak{R}^{\mathbf{44}}(u) = u\mathbb{I} + \mathbb{P} \qquad \text{and} \qquad \mathfrak{R}^{\mathbf{4\bar{4}}}(u) = -(u+2+\alpha)\mathbb{I} + \mathbb{K} \tag{3.4}$$

for an arbitrary constant α. Here, we have introduced identity operator $\mathbb{I}$, trace operator $\mathbb{K}$, and permutation operator $\mathbb{P}$:

$$(\mathbb{I}_{k\ell})^{I_k I_\ell}_{J_k J_\ell} = \delta^{I_k}_{J_k}\delta^{I_\ell}_{J_\ell} \qquad (\mathbb{K}_{k\ell})^{I_k I_\ell}_{J_k J_\ell} = \delta^{I_k I_\ell}\delta_{J_k J_\ell} \qquad (\mathbb{P}_{k\ell})^{I_k I_\ell}_{J_k J_\ell} = \delta^{I_k}_{J_\ell}\delta^{I_\ell}_{J_k}\,, \tag{3.5}$$

acting as braiding operations mapping tensor product vector space $V_k \otimes V_\ell$ to itself.

We also need to construct another set of R-matrices $\mathfrak{R}^{\mathbf{\bar{4}\bar{4}}}(u)$ and $\mathfrak{R}^{\mathbf{\bar{4}4}}(u)$ generating another alternative spin chain system. We again require them to fulfill the respective Yang-Baxter equations:

$$\mathfrak{R}^{\mathbf{\bar{4}\bar{4}}}_{12}(u-v)\,\mathfrak{R}^{\mathbf{\bar{4}\bar{4}}}_{13}(u)\,\mathfrak{R}^{\mathbf{\bar{4}\bar{4}}}_{23}(v) = \mathfrak{R}^{\mathbf{\bar{4}\bar{4}}}_{23}(v)\,\mathfrak{R}^{\mathbf{\bar{4}\bar{4}}}_{13}(u)\,\mathfrak{R}^{\mathbf{\bar{4}\bar{4}}}_{12}(u-v) \tag{3.6}$$
$$\mathfrak{R}^{\mathbf{44}}_{12}(u-v)\,\mathfrak{R}^{\mathbf{\bar{4}4}}_{13}(u)\,\mathfrak{R}^{\mathbf{\bar{4}4}}_{23}(v) = \mathfrak{R}^{\mathbf{\bar{4}4}}_{23}(v)\,\mathfrak{R}^{\mathbf{\bar{4}4}}_{13}(u)\,\mathfrak{R}^{\mathbf{44}}_{12}(u-v) \tag{3.7}$$

Again, we find that the R-matrices that solve (3.6), (3.7) are given by

$$\mathfrak{R}^{\mathbf{\bar{4}\bar{4}}}(u) = u\mathbb{I} + \mathbb{P} \qquad \text{and} \qquad \mathfrak{R}^{\mathbf{\bar{4}4}}(u) = -(u+2+\bar{\alpha})\mathbb{I} + \mathbb{K}\,, \tag{3.8}$$

where $\bar{\alpha}$ is an arbitrary constant.

In the two sets of Yang-Baxter equations, the constants $\alpha, \bar{\alpha}$ are undetermined. We shall now restrict them by requiring unitarity. The unitarity of the combined spin chain system sets the following conditions:

$$\begin{aligned} \mathfrak{R}^{\mathbf{44}}(u)\,\mathfrak{R}^{\mathbf{44}}(-u) \ &= \rho(u)\mathbb{I} \\ \mathfrak{R}^{\mathbf{\bar{4}\bar{4}}}(u)\,\mathfrak{R}^{\mathbf{\bar{4}\bar{4}}}(-u) \ &= \bar{\rho}(u)\ \mathbb{I} \\ \mathfrak{R}^{\mathbf{4\bar{4}}}(u)\,\mathfrak{R}^{\mathbf{\bar{4}4}}(-u) \ &= \sigma(u)\ \mathbb{I} \end{aligned} \tag{3.9}$$

[7]For a construction in SU(3), see [29]. Generalizations to arbitrary Lie (super)algebras and quantum deformations thereof were studied in [27]–[30].

where $\rho(u) = \rho(-u), \bar{\rho}(u) = \bar{\rho}(-u), \sigma(u)$ are c-number functions. It is simple to show that the first two unitarity conditions are indeed satisfied for any $\alpha, \bar{\alpha}$. It is equally simple to show that the last unitarity condition is is satisfied only if $\alpha = -\bar{\alpha}$. Without loss of generality, in what follows, we shall set $\alpha = -\bar{\alpha} = 0$.

Viewing (3.1) as $2L$ sites in a row, we introduce one transfer T-matrix

$$T_0(u,a) = \mathfrak{R}_{01}^{\mathbf{44}}(u)\mathfrak{R}_{02}^{\mathbf{4\bar{4}}}(u+a)\mathfrak{R}_{03}^{\mathbf{44}}(u)\mathfrak{R}_{04}^{\mathbf{4\bar{4}}}(u+a)\cdots\mathfrak{R}_{02L-1}^{\mathbf{44}}(u)\mathfrak{R}_{02L}^{\mathbf{4\bar{4}}}(u+a)\,, \quad (3.10)$$

for one alternate chain and the other T-matrix

$$\overline{T}_0(u,\bar{a}) = \mathfrak{R}_{01}^{\mathbf{\bar{4}4}}(u+\bar{a})\mathfrak{R}_{02}^{\mathbf{\bar{4}\bar{4}}}(u)\mathfrak{R}_{03}^{\mathbf{\bar{4}4}}(u+\bar{a})\mathfrak{R}_{03}^{\mathbf{\bar{4}\bar{4}}}(u)\cdots\mathfrak{R}_{02L-1}^{\mathbf{\bar{4}4}}(u+\bar{a})\mathfrak{R}_{02L}^{\mathbf{\bar{4}\bar{4}}}(u)\,, \quad (3.11)$$

for the other alternate chain, where we introduce an auxiliary zeroth space. By the standard 'train' argument, one can show that the transfer matrices fulfill the Yang-Baxter equations,

$$\mathfrak{R}_{00'}^{\mathbf{44}}(u-v)T_0(u,a)T_{0'}(v,a) = T_{0'}(v,a)T_0(u,a)\mathfrak{R}_{00'}^{\mathbf{44}}(u-v)\,, \quad (3.12)$$

and

$$\mathfrak{R}_{00'}^{\mathbf{\bar{4}\bar{4}}}(u-v)\overline{T}_0(u,\bar{a})\overline{T}_{0'}(v,\bar{a}) = \overline{T}_{0'}(v,\bar{a})\overline{T}_0(u,\bar{a})\mathfrak{R}_{00'}^{\mathbf{\bar{4}\bar{4}}}(u-v)\,. \quad (3.13)$$

In addition, by a similar argument, one may verify that

$$\mathfrak{R}_{00'}^{\mathbf{4\bar{4}}}(u-v+a)T_0(u,a)\overline{T}_{0'}(v,-a) = \overline{T}_{0'}(v,-a)T_0(u,a)\mathfrak{R}_{00'}^{\mathbf{4\bar{4}}}(u-v+a)\,. \quad (3.14)$$

We also define the trace of the T matrix by

$$\tau^{\text{alt}}(u,a) = \operatorname*{Tr}_0 T_0(u,a)\,. \quad (3.15)$$

and

$$\overline{\tau}^{\text{alt}}(u,\bar{a}) = \operatorname*{Tr}_0 \overline{T}_0(u,\bar{a}) \quad (3.16)$$

where the trace is taken over an auxiliary zeroth space.

It then follows from the Yang-Baxter equations that

$$\begin{aligned}
&[\tau^{\text{alt}}(u,a), \tau^{\text{alt}}(v,a)] = 0 \\
&[\overline{\tau}^{\text{alt}}(u,\bar{a}), \overline{\tau}^{\text{alt}}(v,\bar{a})] = 0\,,
\end{aligned} \quad (3.17)$$

and

$$[\tau^{\text{alt}}(u,a), \overline{\tau}^{\text{alt}}(v,-a)] = 0\,. \quad (3.18)$$

Here, in the first two equations, $a, \bar{a}$ are arbitrary and denote two undetermined spectral parameters. These parameters are restricted further if we demand the last equation to hold. Indeed, the two alternating transfer matrices commute each other if and only if $\bar{a} = -a$.

As for all other conserved charges, the Hamiltonian is obtained[8] by evolving the transfer T-matrix infinitesimally in spectral parameter u: $H = \mathrm{d}\log\tau(u,a)|_{u=0}$ where $\mathrm{d} \equiv \partial/\partial u$. By a straightforward computation, we obtain the $\mathbf{4\overline{4}}$ spin chain Hamiltonian as

$$\begin{aligned} H_{2\ell-1} &= -(2-a)\mathbb{I} - (4-a^2)\mathbb{P}_{2\ell-1,2\ell+1} \\ &\quad -(a-2)\mathbb{P}_{2\ell-1,2\ell+1}\mathbb{K}_{2\ell-1,2\ell} + (a+2)\mathbb{P}_{2\ell-1,2\ell+1}\mathbb{K}_{2\ell,2\ell+1}\,, \end{aligned} \tag{3.19}$$

where we scaled the Hamiltonian by multiplying (a^2-4).

By the same procedure, we also find that the Hamiltonian for for the $\mathbf{\overline{4}4}$ spin chain is given by

$$\begin{aligned} \overline{H}_{2\ell} &= -(2+a)\mathbb{I} - (4-a^2)\mathbb{P}_{2\ell,2\ell+2} \\ &\quad +(a+2)\mathbb{P}_{2\ell,2\ell+2}\mathbb{K}_{2\ell,2\ell+1} - (a-2)\mathbb{P}_{2\ell,2\ell+2}\mathbb{K}_{2\ell+1,2\ell+2}\,, \end{aligned} \tag{3.20}$$

where we have replaced $\bar{a}$ by a using the relation $\bar{a} = -a$.

At this stage, any choice of the parameter a is possible in so far as hermiticity of the Hamiltonian is satisfied. The latter condition requires that a is a pure imaginary number. Physically, we are interested in the situation where $\mathbf{4} \leftrightarrow \mathbf{\overline{4}}$ is a symmetry. This is nothing but requiring charge conjugation symmetry, equivalently, reflection symmetry in dual lattice. We thus put $a = i0$[9] Adding the two alternate Hamiltonians, we get total Hamiltonian:[10]

$$H_{\text{total}} = \sum_{\ell=1}^{2L} H_{\ell,\ell+1,\ell+2} \tag{3.22}$$

with

$$H_{\ell,\ell+1,\ell+2} = \Big[4\mathbb{I} - 4\mathbb{P}_{\ell,\ell+2} + 2\mathbb{P}_{\ell,\ell+2}\mathbb{K}_{\ell,\ell+1} + 2\mathbb{P}_{\ell,\ell+2}\mathbb{K}_{\ell+1,\ell+2}\Big]\,. \tag{3.23}$$

In this derivation, there is always a freedom of shifting ground state energy by an arbitrary constant. From the outset, we assumed integrability but, except that the symmetry algebra is $\mathrm{SU}_R(4)$ and that spins are $\mathbf{4}, \mathbf{\overline{4}}$ at alternating lattice sites, we did not utilize any inputs from underlying supersymmetry. With extra input that that supersymmetric ground-state has zero energy, one can always fix the freedom. The (3.23) is the Hamiltonian after being shifted by $+6$ per site accordingly.

[8] The following derivation of Hamiltonian is valid only for $L \geq 2$. This means that the energy eigenvalues of the following Hamiltonians for the case $L = 1$ do not agree with true energy eigenvalues.

[9] Alternatively, one may relax hermiticity of the Hamiltonian and only demand symmetry under parity and time-reversal, leading to so-called PT-symmetric system [41]. This again sets a to zero. Strictly speaking, however, this latter condition is weaker than the hermiticity requirement.

[10] We remark the following useful identities

$$\mathbb{P}_{\ell,\ell+2}\mathbb{K}_{\ell,\ell+1} = \mathbb{K}_{\ell+1,\ell+2}\mathbb{P}_{\ell,\ell+2}, \qquad \mathbb{P}_{\ell,\ell+2}\mathbb{K}_{\ell+1,\ell+2} = \mathbb{K}_{\ell,\ell+1}\mathbb{P}_{\ell,\ell+2}\,. \tag{3.21}$$

We shall find them useful later when investigating issues concerning wrapping interactions.

4. Integrable spin chain from Chern-Simons

In this section, we approach integrability from weak ‘t Hooft coupling regime of the superconformal Chern-Simons theory. We use perturbation theory and look for a spin chain Hamiltonian as a quantum part of the dilatation operator acting on the single trace operators. As mentioned above, in three-dimensional spacetime, general power-counting indicates that logarithmic divergence arises only at even loop orders. Therefore, leading-order contribution to anomalous dimension starts at two loops. In general, as well understood from general considerations of the renormalization theory, the divergence in one-particle irreducible diagrams with one insertion of a composite operator contain divergences that are proportional to other composite operators. Therefore, at each order in perturbation theory, all composite operators whose divergences are intertwined must be renormalized simultaneously. In addition, renormalization of elementary fields needs to be taken into account. This leads to the general structure:

$$\mathcal{O}^M_{\text{bare}}(Y_{\text{bare}}, Y^\dagger_{\text{bare}}) = \sum_N Z^M{}_N \mathcal{O}^N_{\text{ren}}(ZY_{\text{ren}}, ZY^\dagger_{\text{ren}}) \tag{4.1}$$

For the operators we are interested in, this takes the form of

$$\mathcal{O}^M_{\text{bare}} = \sum_N Z^M{}_N(\Lambda)\mathcal{O}^N_{\text{ren}} \tag{4.2}$$

with the UV cut-off scale Λ. Therefore, the anomalous dimension matrix Δ is given by

$$\Delta = \frac{\mathrm{d}\log Z}{\mathrm{d}\log\Lambda}. \tag{4.3}$$

In the rest of this section, we compute anomalous dimension matrix for the single trace operators that were associated with the ‘alternating spin chain’ in the last section:

$$\mathcal{O}^{(I)}_{(J)} = \mathrm{Tr}\left(Y^{I_1}Y^\dagger_{J_1}Y^{I_2}Y^\dagger_{J_2}\cdots Y^{I_L}Y^\dagger_{J_L}\right). \tag{4.4}$$

In $\mathcal{N}=6$ superconformal Chern-Simons theory, the scalar fields $Y^I, Y^\dagger_I$ are bifundamental fields of $\mathrm{U}(N)\times\overline{\mathrm{U(N)}}$ gauge group, and transform as $\mathbf{4}$ and $\overline{\mathbf{4}}$ of $\mathrm{SU}_R(4)$ R-symmetry group. In appendix A, we explain field contents and action of the theory in detail.[11] Schematically, the action of the ABJM theory takes the form

$$I = \int_{\mathbb{R}^{2,1}} \frac{k}{4\pi}\Big(\mathrm{CS}(A) - \mathrm{CS}(\overline{A})\Big) - \mathrm{Tr}(DY)^\dagger_I DY^I + \mathrm{Tr}\ \Psi^{I\dagger} i\not{D}\Psi_I - V_{\mathrm{F}} - V_{\mathrm{B}}. \tag{4.5}$$

Here, the Chern-Simons density is given by

$$\mathrm{CS}(A) = \epsilon^{mnp}\,\mathrm{Tr}\left[A_m\partial_n A_p + \frac{2i}{3}A_m A_n A_p\right]. \tag{4.6}$$

Covariant derivatives are denoted as D_m, while self-interactions involving bosons and fermion pairs are denoted by $V_{\mathrm{B}}, V_{\mathrm{F}}$, respectively. See appendix A for their explicit form. We will recall them at relevant points in foregoing discussions.

[11] We closely follow notation and convention of [42].

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To extract the dilatation operator, we compute the correlation functions

$$\left\langle \mathcal{O}^{(I)}_{(J)} \mathrm{Tr}(Y^\dagger_{I_1} Y^{J_1} \cdots Y^\dagger_{I_L} Y^{J_L}) \right\rangle \qquad \text{for} \qquad L \to \infty \tag{4.7}$$

by summing over all planar diagrams in perturbation theory in 't Hooft coupling λ.

In evaluating so, there arises an important issue regarding consistency of regularization with gauge invariance and $\mathcal{N} = 6$ supersymmetry. We shall adopt dimensional reduction method (See, for example, discussions in [43]). This method retains ϵ^{mnp} and Dirac matrices always three-dimensional. In each Feynman integral, we then manipulate the integrand until all ϵ^{mnp} and Dirac matrices are eliminated and the integral is reduced to a Lorentz scalar expression. We then employ dimensional regularization and evaluate the integral. Still, this leaves out infrared divergences that would have been absent were if the theory four-dimensional. As we will be only concerned with logarithmic ultraviolet divergences, we will take a practical approach that we regularize infrared divergences by introducing mass terms in evaluating Feynman integrals in dimensional regularization. We then remove the regulator mass first and then take the spacetime dimension to three. Previously, it was checked that the dimensional reduction method is consistent with Slavnov-Taylor-Ward identities. Yet, to date, it is not known if the method is compatible with $\mathcal{N} = 6$ supersymmetry. Thus, in our computations, we shall not assume a priori any input related to supersymmetry. Rather, we will put our result to a test against various consequences of supersymmetry — for instance, vanishing anomalous dimensions of chiral primary operators and superconformal nonrenormalization theorems.

Using the convention and Feynman rules explained in appendix, we computed all two-loop diagrams that contribute to anomalous dimensions of elementary fields $Y^I, Y^\dagger_I$ and composite operators $\mathcal{O}^{(I)}_{(J)}$. Acting on the space of the operators, each Feynman diagram can be attributed to the braiding operations $\mathbb{I}$, $\mathbb{K}$, $\mathbb{P}$ introduced in (3.5) and their combinations. At two loops, we computed the complete set of Feynman diagrams that contribute to each of these operators. The result turned out

$$H_{2-\text{loops}} = \lambda^2 \sum_{\ell=1}^{2L} \left[\mathbb{I} - \mathbb{P}_{\ell,\ell+2} + \frac{1}{2}\mathbb{P}_{\ell,\ell+2}\mathbb{K}_{\ell,\ell+1} + \frac{1}{2}\mathbb{P}_{\ell,\ell+2}\mathbb{K}_{\ell+1,\ell+2} \right] \tag{4.8}$$

and this is precisely $\frac{\lambda^2}{4}$ times the alternating spin chain Hamiltonian (3.23) we derived from SU(4) Yang-Baxter equations in the last section. In the rest of this section, we explain essential steps for deriving the Hamiltonian and relegate technical details of evaluating Feynman diagrams in the appendix. We find it convenient to organize contributing Feynman diagrams according to the number of sites that participate in the Hamiltonian.

Three-site scalar interactions. A salient feature of the alternating spin chain Hamiltonian we extracted in section 3 from coupled Yang-Baxter equations is that it contains interactions up to next-nearest-neighbor sites. We thus need to see if such interaction arises from superconformal Chern-Simons planar diagrams and, if so, if the interactions are of the same type. From the Feynman rules (see appendix A), it is evident that scalar interaction

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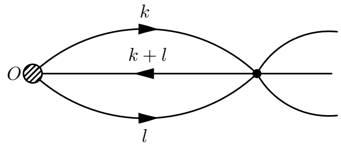

Figure 1: Two loop contribution of scalar sextet interaction to anomalous dimension of $\mathcal{O}$.

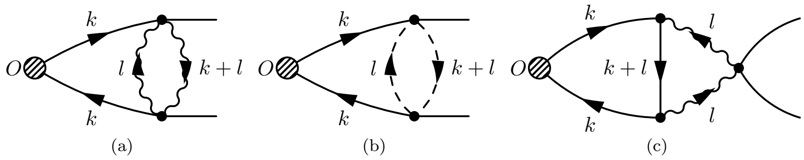

Figure 2: Two loop contribution of gauge and fermion exchange interaction to anomalous dimension of $\mathcal{O}$.

$-V_{\rm B}$ in (4.5) is *the* source of three-site interactions, whose explicit form is given by

$$V_{\rm B} = -\frac{1}{3}\left(\frac{2\pi}{k}\right)^2 \overline{\rm Tr}\Big[Y_I^\dagger Y^J Y_J^\dagger Y^K Y_K^\dagger Y^I + Y_I^\dagger Y^I Y_J^\dagger Y^J Y_K^\dagger Y^K \\ +4 Y_I^\dagger Y^J Y_K^\dagger Y^I Y_J^\dagger Y^K - 6 Y_I^\dagger Y^I Y_J^\dagger Y^K Y_K^\dagger Y^J \Big] \qquad (4.9)$$

The two-loop Feynman diagram is depicted in figure 1. From planar diagram combinatorics of gauge invariant operators at infinite length $2L \to \infty$, we find the following contributions arising: $\mathbb{K}_{\ell,\ell+1} + \mathbb{K}_{\ell+1,\ell+2}$ from the first two terms, $\mathbb{P}_{\ell,\ell+2}$ from the third term, and $\mathbb{I} + \mathbb{P}_{\ell,\ell+2}\mathbb{K}_{\ell,\ell+1} + \mathbb{P}_{\ell,\ell+2}\mathbb{K}_{\ell+1,\ell+2}$ from the last term. Taking account of combinatorial multiplicities, we find that the scalar sextet potential contributes to the dilatation Hamiltonian as

$$H_{\rm B} = \lambda^2 \sum_{\ell=1}^{2L} \left[\frac{1}{2}\mathbb{I} - \mathbb{P}_{\ell,\ell+2} + \frac{1}{2}\mathbb{P}_{\ell,\ell+2}\mathbb{K}_{\ell,\ell+1} + \frac{1}{2}\mathbb{P}_{\ell,\ell+2}\mathbb{K}_{\ell+1,\ell+2} - \frac{1}{2}\mathbb{K}_{\ell,\ell+1} \right] \qquad (4.10)$$

(see appendix B2). Evidently, compared to the anticipated alternating spin chain Hamiltonian, we have discrepancy in on-site (proportional to $\mathbb{I}$) and nearest neighbor (proportional to $\mathbb{K}$) terms. These are interactions that would arise from gauge or fermion-pair exchange interactions and from wave function renormalization of elementary fields $Y, Y^\dagger$.

Two-site gauge and fermion interactions. The scalar fields $Y^I, Y_I^\dagger$ are bifundamentals of $\mathrm{U}(N) \times \overline{\mathrm{U(N)}}$. Their gauge interactions can be read off from covariant derivatives:

$$D_m Y^I = \partial_m Y^I + i A_m Y^I - i Y^I \overline{A}_m \quad \text{and} \quad D_m Y_I^\dagger = \partial_m Y_I^\dagger + i\overline{A}_m Y_I^\dagger - i Y_I^\dagger A_m\,. \qquad (4.11)$$

As usual, there are paramagnetic interactions (minimal coupling) and diamagnetic interactions (seagull coupling). We see that gauge interactions contribute to two-site terms for both $\mathbb{I}$ and $\mathbb{K}$. Two relevant Feynman diagrams are (a) and (c) in figure 2.

The Feynman diagram contributing to $\mathbb{I}$ operator arises from square of diamagnetic interactions in t-channel. See figure 2(a). This diagram is infrared divergent for each subgraphs. We regulate them by giving a mass to internal propagators. Upon removing the regulator mass to zero, we find a finite part. However, this part turned out ultraviolet convergent and hence does not contribute to anomalous dimension. The Feynman diagram contributing to $\mathbb{K}$ operator arises from product of diamagnetic interaction and two paramagnetic interactions. See figure 2(c). Taking the net momentum of $\mathcal{O}$ to zero, which is sufficient for extracting anomalous dimension, we find that only one orientation of diamagnetic interaction vertex yields nonvanishing result. For details of Feynman rules of gauge interactions and Feynman diagram evaluation, see appendix B3. We found that gauge interactions contribute to the dilatation operator by

$$H_{\rm gauge} = \lambda^2 \sum_{\ell=1}^{2L} \Big[-\frac{1}{4}\mathbb{I} - \frac{1}{2}\mathbb{K}_{\ell,\ell+1} \Big]. \tag{4.12}$$

Consider next two-site terms induced by fermion-pair exchange diagrams. The relevant part of the Lagrangian in (4.5) is the fermion-pair potential:

$$\begin{aligned} V_{\rm F} = \ & \frac{2\pi i}{k}\mathrm{Tr}\Big[Y_I^\dagger Y^I \Psi^{\dagger J}\Psi_J - 2Y_I^\dagger Y^J \Psi^{\dagger I}\Psi_J + \epsilon^{IJKL} Y_I^\dagger \Psi_J Y_K^\dagger \Psi_L\Big] \\ & - \frac{2\pi i}{k}\mathrm{Tr}\Big[Y^I Y_I^\dagger \Psi_J \Psi^{\dagger J} - 2Y^I Y_J^\dagger \Psi_I \Psi^{\dagger J} + \epsilon_{IJKL} Y^I \Psi^{\dagger J} Y^K \Psi^{\dagger L}\Big] \,. \end{aligned} \tag{4.13}$$

From Feynman rules, we see that planar diagrams formed by square of the second terms in both lines in (4.13) give rise to $\mathbb{K}$ interactions to the two-loop dilatation operator. See figure 2(b) for the relevant Feynman diagram and appendix B3 for the details of computation.

In fact, at planar approximation, there is no other Feynman diagrams that contribute to two-site interactions.[12] Taking account of numerical weights in (4.13), we find that the fermion potential contributes to the dilatation Hamiltonian as

$$H_{\rm F} = \lambda^2 \sum_{\ell=1}^{2L} \mathbb{K}_{\ell,\ell+1} \,. \tag{4.14}$$

One-site interactions: wave function renormalization. Adding up all the two-site interactions to the three-site interaction, we see that terms involving $\mathbb{K}$ operator cancel out one another. On the other hand, terms involving $\mathbb{I}$ operator add up to $(1/4)\lambda^2$. So, up to overall (volume-dependent) shift of the ground state energy, the dilatation operator agrees with the alternating spin chain Hamiltonian we derived in the previous section. As we are dealing with superconformal field theory, spectrum of dilatation generator bears an absolute meaning. Moreover, there could be potential clash between dimensional reduction we used and superconformal invariance. Therefore, to ensure internal consistency of quantum theory, we shall now compute terms arising from wave function renormalization of $Y, Y^\dagger$. These are all the remaining contributions to anomalous dimension of composite operator $\mathcal{O}$.

Wave function renormalization to $Y, Y^\dagger$ arises from all three types of interactions. Even though there are huge numbers of planar Feynman diagrams that could potentially

[12] For $L = 1$, however, there will be wrapping interactions. We will discuss them in detail in the next section.

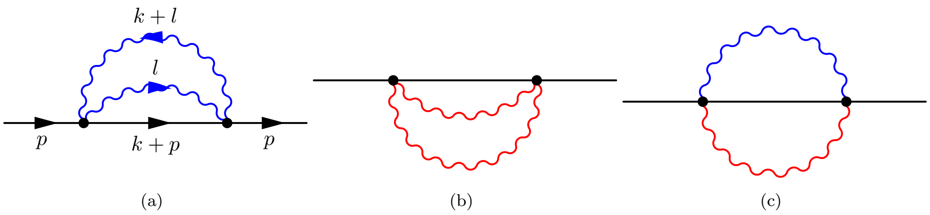

Figure 3: Two loop contribution of diamagnetic gauge interactions to wave function renormalization of $Y, Y^\dagger$. They contribute to $\mathbb{I}$ operator in the dilatation operator.

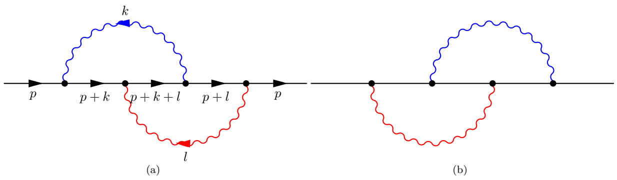

Figure 4: Two loop contribution of paramagnetic gauge interactions to wave function renormalization of $Y, Y^\dagger$. They contribute to $\mathbb{I}$ operator in the dilatation operator.

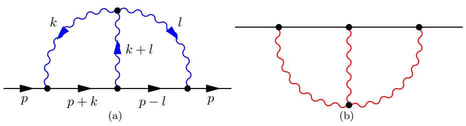

Figure 5: Two loop contribution of Chern-Simons interaction to wave function renormalization of $Y, Y^\dagger$. They contribute to $\mathbb{I}$ operators in the dilatation operator.

contribute to wave function renormalization, a vast number of them vanishes identically or cancel one another. First, diagrams involving gauge boson loops either vanish because of parity-odd nature of the gauge boson propagators or cancel among $\mathrm{U}(N)$ and $\overline{\mathrm{U(N)}}$ diagrams.[13] Nonzero contribution arise only from diamagnetic interactions shown in figure 3, from paramagnetic interactions shown in figure 4, and from Chern-Simons cubic interactions shown in figure 5.

Second, diagrams involving vertices in the first and the second lines in V_F (4.13) cancel by combinatorics and relative coefficients. Hence, the cancellation is attributable to $\mathcal{N}=6$ supersymmetry. The only surviving diagram arise from cross term of vertices in the last line in (4.13). The Feynman diagram is shown in figure 6.

Third, there are also contributions coming from gauge-matter interactions. Again, almost all diagrams vanish because of parity-odd nature of gauge boson propagator. The only surviving diagrams involve parity-even vacuum polarization, as shown in figure 7. Their computations are summarized in appendix B4. We also present the analysis of the one-loop vacuum polarizations in appendix B5.

[13]Notice that gauge boson propagator for $\mathrm{U}(N)$ and $\overline{\mathrm{U(N)}}$ gauge groups have weight $+k$ and $-k$, respectively.

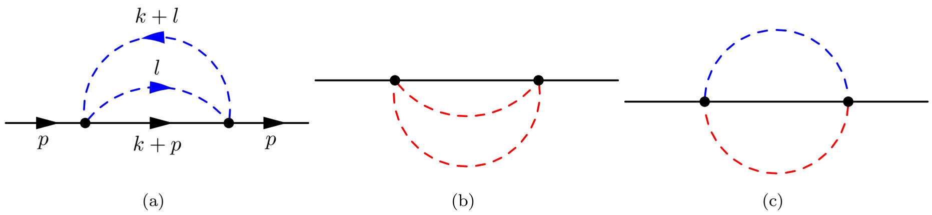

Figure 6: Two loop contribution of fermion pair interaction to wave function renormalization of $Y, Y^\dagger$. They contribute to $\mathbb{I}$ operators in the dilatation operator.

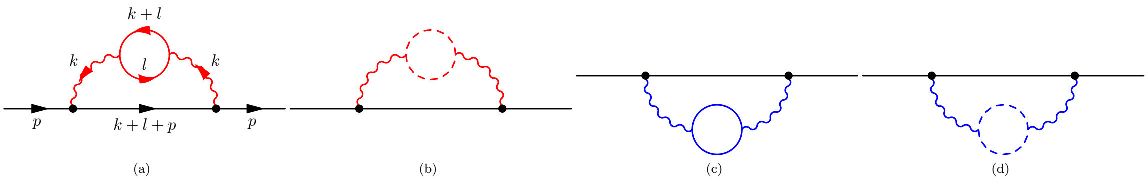

Figure 7: Two loop contribution of vacuum polarization to wave function renormalization of $Y, Y^\dagger$. Both $\mathrm{U}(N)$ and $\overline{\mathrm{U(N)}}$ gauge parts give additive contributions.

Summing up all these wave function renormalization to $Y, Y^\dagger$, we find their contribution to the dilatation operator as

$$\begin{aligned} H_{\mathrm{Z}} &= \lambda^2 \left[\left(\frac{1}{12} + \frac{2}{3} + \frac{1}{3} \right) + \left(\frac{4}{3} + 1 \right) - \frac{8}{3} \right] \sum_{\ell=1}^{2L} \mathbb{I} \\ &= \lambda^2 \sum_{\ell=1}^{2L} \frac{3}{4} \mathbb{I} \end{aligned} \tag{4.15}$$

In the first line, the first parenthesis is the contribution from gauge fields: diamagnetic interactions, paramagnetic interactions, and Chern-Simons interactions. The second parenthesis is the contribution from fermion fields. The last term is the contribution of vacuum polarization. Adding up all the contributions,

$$H_{\mathrm{total}} = H_{\mathrm{B}} + H_{\mathrm{F}} + H_{\mathrm{gauge}} + H_{\mathrm{Z}} \tag{4.16}$$

we get the result (4.8). As claimed, this is precisely the alternating spin chain Hamiltonian we obtained from mixed set of Yang-Baxter equations. As such, we conclude that dilatation operator of $\mathcal{N} = 6$ superconformal Chern-Simons theory of ABJM is integrable at two loops.

We stress the importance of explicit and direct computation of the dilatation operator without a prior assumption relying on supersymmetry or integrability. It is satisfying that the result passes various compatibility tests. For instance, take chiral primary operators. These are subset of the single trace operators $\mathcal{O}$ where Y's and $Y^\dagger$'s are totally symmetric and traceless under any contraction between Y's and $Y^\dagger$'s, and corresponds to massive Kaluza-Klein modes over $\mathbb{CP}^3$ in the Type IIA supergravity dual. Because of supersymmetry, their scaling dimension should be protected against radiative corrections. Indeed, acting on these operators, $H_{\rm total}$ vanishes since contribution of terms involving $\mathbb{K}$ operator are null and contribution of $\mathbb{P}$ cancel against that of $\mathbb{I}$. As a corollary, the fact that our result is consistent with expectation from supergravity dual implies that the dimension reduction method we adopted for computations are compatible not only with Slavnov-Taylor identities of the gauge symmetry but also with $\mathcal{N}=6$ supersymmetry.

5. The shortest chain and wrapping interactions

In deriving the dilatation operator in the last section, we assumed that the gauge invariant operator is infinitely long, $L \to \infty$. From planar diagrammatics, we see easily that dilatation operator computed perturbatively up to the order 2ℓ will give rise to a spin chain Hamiltonian whose range extends to (2ℓ)-th order. Therefore, for operators of finite length, a new set of planar diagrams which wraps around the operator will come in to contribute. These are so-called wrapping interactions, a feature discussed much in the context of integrability of four-dimensional $\mathcal{N}=4$ super Yang-Mills theory [12, 44]–[53].

In $\mathcal{N}=6$ superconformal Chern-Simons theory, the situation is more interesting. Since the dilatation operator at two loops ranges over three sites, spectrum of the shortest gauge invariant operator of length $2L=2$ will receive contributions from wrapping diagrams already at leading order! In this section, we like to identify these wrapping interactions for the shortest gauge invariant operators and discuss their implications.

Let us denote basis of the shortest operators as

$$|I_1 I_2\rangle = \operatorname{Tr} Y^{I_1} Y^\dagger_{I_2} = O_{I_1 I_2} \in \mathbf{4} \otimes \overline{\mathbf{4}}\,. \tag{5.1}$$

The $\mathbf{4}\otimes\overline{\mathbf{4}}$ representation is decomposed irreducibly into the traceless part, $\mathbf{15}$, and the trace part, $\mathbf{1}$. The multiplet $\mathbf{15}$ is chiral primary operator, so their conformal dimension ought to be protected by supersymmetry.

To check this, let us first identify the two-site dilatation operator that includes the wrapping interactions. At two-loop orders, the scalar sextet interaction does not contribute to length-$2L=2$ operators since only four legs can be connected to the operators, leaving a tadpole that vanishes identically. Hence the dilatation operator consists of the two-site plus wave function renormalization parts plus wrapping contributions.

From the computations of section 4, the original two-site contributions comprise of two-loop diagrams from gauge interactions and from $V_{\rm F}$ interactions. Their contributions are

$$H_2 = \left[\left(-\frac{1}{2}\mathbb{K} - \frac{1}{4}\mathbb{I}\right)\lambda^2 + \mathbb{K}\lambda^2\right] \times 2 = \left(\mathbb{K} - \frac{1}{2}\mathbb{I}\right)\lambda^2\,. \tag{5.2}$$

In the first line, the first term is the contribution of gauge interaction diagrams and the second term is the contribution of $V_{\rm F}$ interactions. We computed total energy, so multiplied the energy density by the spin chain volume $2L=2$. The one-site contribution arising from the wave function renormalization is

$$H_1 = \frac{3}{4}\lambda^2\,\mathbb{I} \times 2 = \frac{3}{2}\lambda^2\,\mathbb{I}\,. \tag{5.3}$$

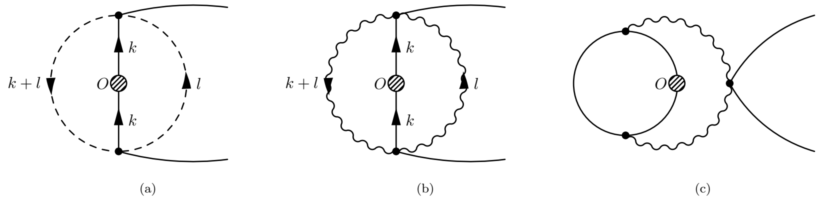

Figure 8: Two loop wrapping interaction contribution to the shortest gauge invariant operators. (a) fermion field wrapping, (b) gauge field wrapping, (c) a new gauge triangle.

Adding these two and acting on **15** in (5.1), we see that anomalous dimension of the chiral primary operator is non-vanishing. If our regularization method of dimensional reduction plus infrared mass regularization were compatible with supersymmetry, there must be other contributions heretofore unaccounted that would cancel against the non-vanishing contribution (5.3) and protect the anomalous dimension of chiral primary operator from quantum corrections. These are precisely wrapping interactions.

Indeed, for the shortest operators of $L = 1$ under consideration, there are three classes of nontrivial wrapping interactions. We now summarize their contribution and relegate details of Feynman diagram evaluation to appendix C.

There is the gauge field wrapping contribution with the diamagnetic interactions as in figure 8 (a). Its contribution is

$$H_{gIw} = \lambda^2 \mathbb{I}\,. \tag{5.4}$$

There is also the fermion field wrapping contribution as in figure 8 (b). Its contribution is

$$H_{yw} = 2\,(\mathbb{K} - \mathbb{I})\,\lambda^2\,. \tag{5.5}$$

It is important to note that these two wrapping interactions utilizes simultaneously $\mathrm{U}(N)$ and $\overline{\mathrm{U(N)}}$ interactions. Thus, this contribution arises not just by distinct topology of planar diagram but from very different interactions from the original, unwrapped two-site interactions.

There is also a doubling-type wrapping contribution of using the same gauge group interactions. This happens only for the gauge interaction diagram contributing to $\mathbb{K}$ operator. Moreover, the contribution is doubled since there are two distinct ways of wrapping. This is best illustrated on a cylinder, from which we see that there are two different kinds of topology of wrapped Feynman diagrams. From appendix C, we identify this contribution as

$$H_{gKw} = -\lambda^2\,\mathbb{K}\,. \tag{5.6}$$

Putting both the original and the wrapping diagram contributions together, the full Hamiltonian of $2L = 2$ operator is given by

$$H_{2L=2} = 2\,\lambda^2\,\mathbb{K}\,. \tag{5.7}$$

Notice that the part proportional to $\mathbb{I}$ operator is canceled between the original and the wrapping interaction contributions. One thus check that the chiral primary operators **15**

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indeed has a vanishing anomalous dimension since, by definition, it has no trace part and is annihilated by $\mathbb{K}$ operator. For the singlet **1**, $|s\rangle = \frac{1}{2}|II\rangle$, the anomalous dimension is

$$H|s\rangle = 8\,\lambda^2\,|s\rangle\,. \tag{5.8}$$

It is interesting to compare the above spectrum with spectrum of the naive Hamiltonian $H_{\rm naive}$, viz. the alternating spin chain Hamiltonian with periodic boundary condition and $2L = 2$. The latter is[14]

$$H_{\rm naive} = \lambda^2 \sum_{\ell=1}^{2} \Big[\mathbb{I} - \mathbb{P}_{\ell,\ell+2} + \frac{1}{2}\mathbb{K}_{\ell+1,\ell+2}\mathbb{P}_{\ell,\ell+2} + \frac{1}{2}\mathbb{K}_{\ell,\ell+1}\mathbb{P}_{\ell,\ell+2}\Big]_{\ell+2=\ell} = 2\,\lambda^2\,\mathbb{K}\,, \tag{5.9}$$

for $2L = 2$. Acting on **15** and **1** states, we find that their anomalous dimension is 0 and $4 \cdot 2\lambda^2$, respectively. So far, we computed the spectrum of the shortest operators without a priori assumption of supersymmetry. As a consistency check, we now compare these spectra with their superpartners. Recall that length 2ℓ operators with Dynkin labels $(\ell-2m, m+n, \ell-2n)$ and length $2\ell-2$ operators with Dynkin labels $(\ell-2m, m+n-2, \ell-2n)$ are superpartners each other. Here, we have the simplest situation: the $L = 1$ operator **1** of Dynkin labels $(0, 0, 0)$ is the superpartner of $L = 2$ operator **20** of Dynkin labels $(0, 2, 0)$. Fortuitously, anomalous dimension of the latter was computed at two loops in [23] to be $8\lambda^2$, and matches perfectly with our computation.[15] Note that, at two loop order, the $L = 2$ operator **20** does not receive any wrapping interaction corrections. As such, we may consider agreement of the anomalous dimensions between the two superpartners as a nontrivial confirmation for the wrapping interactions we studied for the $L = 1$ operator **1**.

We should also note that the naive Hamiltonian is not the right dilatation operator for the shortest operators. Nevertheless, interestingly, the spectrum of naive Hamiltonian coincides with the spectrum extracted from the true two-site Hamiltonian. It would be very interesting to see whether this coincidence persists to higher orders in perturbation theory.

6. Bethe ansatz diagonalization

In section 3, we constructed transfer matrix. To obtain spectrum, we need to diagonalize the transfer matrices. Within algebraic Bethe ansatz, a fairly general result is known for a Lie (super)groups G [31, 27, 30]. It suffices to adapt the results to the case that $G = \mathrm{SU}(4)$.[16] Dynkin diagram of SU(4), drawn horizontally, has three roots: left(l), middle(m), and right(r). The diagonalization is specified by the choice of Dynkin label (R_l, R_m, R_r) for the site representation R and total number of sites L_R that representation occupies. In the present case, we have placed **4** and $\overline{\mathbf{4}}$ representations at alternating lattices, so $R_l = R_r = 1, R_m = 0$ and $L_{\mathbf{4}} = L_{\overline{\mathbf{4}}} = L$. Each excitation is associated with three sets of Bethe ansatz rapidities (l_a, m_b, r_c)'s whose labels range over $[1, N_l], [1, N_m], [1, N_r]$, respectively. It belongs to the SU(4) representation with the Dynkin labels $(L - 2N_l +$

[14]Here, we used the identities (3.21).

[15]We thank Joe Minahan and Kostya Zarembo for useful correspondences on this issue.

[16]For SU(3) alternating spin chain, this was done explicitly in [29].

$N_m, N_l + N_r - 2N_m, L - 2N_r + N_m)$. Positivity of the Dynkin labels restricts range of the three Bethe ansatz rapidities accordingly. Then, choosing the highest-weight state:

$$|\Omega_+\rangle = \prod_{\ell=1}^{L} \otimes |1\rangle_{2\ell-1} |\overline{4}\rangle_{2\ell} \equiv |1\overline{4}1\overline{4}\cdots\rangle \tag{6.1}$$

as the ground-state, the eigenvalue of the transfer matrix $T_0(-u)$[17] is found to be

$$\begin{aligned}\Lambda(u) \,=\, & (u-1)^L(u-2)^L \prod_{a=1}^{N_l} \frac{u - il_a + \frac{1}{2}}{u - il_a - \frac{1}{2}} + u^L(u-1)^L \prod_{c=1}^{N_r} \frac{u - ir_c - \frac{5}{2}}{u - ir_c - \frac{3}{2}} \\ & + u^L(u-2)^L \left[\prod_{a=1}^{N_l} \frac{u - il_a - \frac{3}{2}}{u - il_a - \frac{1}{2}} \prod_{b=1}^{N_m} \frac{u - im_b - 0}{u - im_b - 1} + \prod_{b=1}^{N_m} \frac{u - im_b - 2}{u - im_b - 1} \prod_{c=1}^{N_r} \frac{u - ir_c - \frac{1}{2}}{u - ir_c - \frac{3}{2}} \right].\end{aligned} \tag{6.2}$$

We have chosen the Bethe rapidities symmetric between the three roots. Keeping the highest weight state the same $|1\overline{4}1\overline{4}\cdots\rangle$, we also find that diagonalization of the second transfer matrix $\overline{T}_0(-v)$ proceeds much the same way as that of $T_0(-v)$ except that we interchange role of the left and the right SU(4) roots:

$$\begin{aligned}\overline{\Lambda}(v) \,=\, & v^L(v-1)^L \prod_{a=1}^{N_l} \frac{v - il_a - \frac{5}{2}}{v - il_a - \frac{3}{2}} + (v-1)^L(v-2)^L \prod_{c=1}^{N_r} \frac{v - ir_c + \frac{1}{2}}{v - ir_c - \frac{1}{2}} \\ & + v^L(v-2)^L \left[\prod_{a=1}^{N_l} \frac{v - il_a - \frac{1}{2}}{v - il_a - \frac{3}{2}} \prod_{b=1}^{N_m} \frac{v - im_b - 2}{v - im_b - 1} + \prod_{b=1}^{N_m} \frac{v - im_b - 0}{v - im_b - 1} \prod_{c=1}^{N_r} \frac{v - ir_c - \frac{3}{2}}{v - ir_c - \frac{1}{2}} \right].\end{aligned} \tag{6.3}$$

Mutually commuting conserved charges are then constructed by expanding these eigenvalues around $u, v = 0$. The first two charges are the total momentum and the total energy:

$$\begin{aligned} P_{\rm total} &= \frac{1}{i}\Big[\log\Lambda(u) + \log\overline{\Lambda}(u)\Big]_{u=0} \\ &= \sum_{a=1}^{N_l} \log\left(\frac{l_a + i/2}{l_a - i/2}\right) + \sum_{b=1}^{N_r} \log\left(\frac{r_b + i/2}{r_b - i/2}\right), \end{aligned} \tag{6.4}$$

$$\begin{aligned} E_{\rm total} &= \lambda^2\Big[\frac{\mathrm{d}}{\mathrm{d}u}(\log\Lambda(u) + \log\overline{\Lambda}(u))\Big]_{u=0} \\ &= \lambda^2\left(\sum_{a=1}^{N_l} \frac{1}{l_a^2 + \frac{1}{4}} + \sum_{b=1}^{N_r} \frac{1}{r_b^2 + \frac{1}{4}}\right). \end{aligned} \tag{6.5}$$

Here, we chose fundamental domain of the momentum to $[0, 2\pi)$ and scaled the total energy by λ^2 in accordance to the relation we fixed between Hamiltonian derived from Yang-Baxter equation and from superconformal Chern-Simons theory. Likewise, we can deduce higher conserved charges from higher moments of the transfer matrices.

The Bethe equations that results from the above transfer matrix eigenvalues $\Lambda, \overline{\Lambda}$ are

$$\left(\frac{l_a - \frac{i}{2}}{l_a + \frac{i}{2}}\right)^L = \prod_{b=1(b\neq a)}^{N_l} \frac{l_a - l_b - i}{l_a - l_b + i} \prod_{c=1}^{N_m} \frac{l_a - m_c + \frac{i}{2}}{l_a - m_c - \frac{i}{2}}$$

[17]For later convenience, we choose to diagonalize T_0 for opposite sign of the spectral parameter u.

$$
\begin{aligned}
1 &= \prod_{b=1(b\neq a)}^{N_m} \frac{m_a - m_b + i}{m_a - m_b - i} \prod_{c=1}^{N_l} \frac{m_a - l_c - \frac{i}{2}}{m_a - l_c + \frac{i}{2}} \prod_{d=1}^{N_r} \frac{m_a - r_d - \frac{i}{2}}{m_a - r_d + \frac{i}{2}} \\
\left(\frac{r_a - \frac{i}{2}}{r_a + \frac{i}{2}}\right)^L &= \prod_{b=1(b\neq a)}^{N_r} \frac{r_a - r_b - i}{r_a - r_b + i} \prod_{c=1}^{N_m} \frac{r_a - m_c + \frac{i}{2}}{r_a - m_c - \frac{i}{2}}. \qquad (6.6)
\end{aligned}
$$

It is straightforward to check that these same set of Bethe ansatz equations remove potential simple pole terms for both Λ and $\overline{\Lambda}$ simultaneously.

From the integrability perspectives, $(2+1)$-dimensional superconformal Chern-Simons theory is quite different from (3+1)-dimensional super Yang-Mills theory. The most distinct feature is that the spin chain associated with dilatation operator is not homogeneous but alternating. It calls for better understanding to questions that arise in comparison with $\mathcal{N} = 4$ super Yang-Mills counterpart. We shall now study spectrum of the Bethe ansatz equations for a few simpler situations and gather features concerning excitations of the alternating spin chain system.

First, consider the special class of $N_m = 0$ for arbitrary $L \geq 2$. The first and the third Bethe ansatz equations decouple, and each equation becomes the same as the Bethe ansatz equation of the well-known SU(2) $\mathrm{XXX}_{\frac{1}{2}}$ spin chain. Thus, if one can identify the first set with SU(2) of $\mathbf{4}$ side, then the third equation corresponds to SU(2) of $\bar{\mathbf{4}}$. We then have two decoupled sets of the solution including towers of bound states, and they are exactly the same as the $\mathrm{XXX}_{\frac{1}{2}}$ spin chain.

Now let us consider the case $N_l = N_m = N_r = 1$ case for a general $L \geq 2$. The Bethe ansatz equations are reduced to

$$
\begin{aligned}
\left(\frac{l - \frac{i}{2}}{l + \frac{i}{2}}\right)^L &= \frac{l - m + \frac{i}{2}}{l - m - \frac{i}{2}} \\
m &= \frac{1}{2}(l + r) \\
\left(\frac{r - \frac{i}{2}}{r + \frac{i}{2}}\right)^L &= \frac{r - m + \frac{i}{2}}{r - m - \frac{i}{2}}. \qquad (6.7)
\end{aligned}
$$

In terms of the individual momentum variables, after using the second equation, the combination of the first and the third equations becomes

$$
e^{i(p_l + p_r)L} = 1\,. \qquad (6.8)
$$

This is solved by

$$
P = p_l + p_r = \frac{2\pi n}{L}, \qquad (n = 0, 1, 2, \dots, L-1)\,. \qquad (6.9)
$$

First, consider $m = 0$ case. In this case, total momentum $P = 0$. For the relative momentum $q \equiv (p_l - p_r)$, we also have

$$
e^{iq(L+1)/2} = 1\,. \qquad (6.10)
$$

This is solved by

$$\frac{q}{2} = \frac{2\pi\mathbb{Z}}{L+1} \tag{6.11}$$

showing that there are in fact $(L + 1)$ independent states. In this case, the energy (6.5) is given by

$$E = 8\lambda^2 \sin^2 \frac{q}{4}. \tag{6.12}$$

This is the simplest example of two-particle excitations where **4** and $\overline{\mathbf{4}}$ excitations are correlated. The total momentum is zero, while total energy depends on relative momentum.

Consider next general m. From the ratio between the first and the third equations, we obtain

$$e^{i\frac{q}{2}L} = \mp \frac{l - r - i}{l - r + i}, \tag{6.13}$$

where we used the second Bethe ansatz equation for simplification. Total momentum P is nonzero. Furthermore, expressing this equation in terms of P and q, we find the relations:

$$\cos\frac{P}{2} = \frac{\sin\frac{q}{4}(L+2)}{\sin\frac{q}{4}L} \qquad \text{or} \qquad \frac{\cos\frac{q}{4}(L+2)}{\cos\frac{q}{4}L}. \tag{6.14}$$

If q is real, viz. two real Bethe roots, the relation shows that relative momentum q is correlated with total momentum P. That is, even though there are two excitations associated with **4** and $\overline{\mathbf{4}}$ chains, their motion exhibits mutual correlation. If q were imaginary, viz. a Bethe string, the relation shows that total momentum ought to be purely imaginary. This show that there cannot arise any bound-state between **4** and $\overline{\mathbf{4}}$ spins. Though this argument is established for the compact SU(4) sector, we expect our conclusion also extends to the noncompact Sp(4) sector.

We can also comment on thermodynamic limit in which densities of the Bethe roots are kept finite. By taking $L \to \infty$ limit of the Bethe ansatz equations and taking the so-called "no hole" excitation condition, we obtain relations among the three Bethe root densities $\rho_l(x), \rho_m(x), \rho_r(x)$. From the first and the third Bethe ansatz equations, after Fourier transform, we find

$$\rho_l(k) = \rho_r(k). \tag{6.15}$$

This has a simple interpretation: because the alternating spin chain is manifestly charge-conjugation invariant, excitations ought to be so as well. Moreover, from the second Bethe ansatz equation, we obtain

$$\rho_m(k)e^{-|k|/2} = \frac{1}{2}\Big[\rho_l(k) + \rho_r(k)\Big]. \tag{6.16}$$

It immediately follows from these two equations that the mean value of root densities

$$\frac{N_l}{L} = \frac{N_r}{L} \qquad \text{and} \qquad \frac{N_m}{L} = \frac{1}{2}\left(\frac{N_l}{L} + \frac{N_r}{L}\right). \tag{6.17}$$

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We conclude that all three Bethe root densities are equal, and hence $\mathbf{4}$'s and $\overline{\mathbf{4}}$'s are equally populated and balanced each other for the minimum energy configuration.

Furthermore, analysis of the shortest operator suggests that excitation in superconformal Chern-Simons spin chain is different from excitation in $\mathcal{N}=4$ super Yang-Mills spin chain. In the latter, the vacuum is ferromagnetic and excitations break $\mathrm{SO}_R(6)$ to $[\mathrm{SU}(2)]^2$. The latter is the symmetry group of dilute, finite-energy excitations. In the present case, analysis of the previous section seems to indicate that excitation is organized by the full SU(4), not by any subgroup of it. This is because the finite energy excitation is a singlet of SU(4), not of any subgroup of it. Lastly, in this system, excitations with $N_m = 0$ comprises of two decoupled $\mathrm{XXX}_{\frac{1}{2}}$ spin chains with its own ferromagnetic vacuum, respectively. Though this is certainly a closed subsector, general excitations in the full system looks quite different, as is seen above in the simple situation of $N_m = 1$.

Following the general prescription [30] and paving the parallels to what was done in the context of $\mathcal{N}=4$ super Yang-Mills theory [8], extending the SU(4) spin chain to the $\mathrm{OSp}(6|;\mathbb{R})$ superspin chain and writing down Bethe ansatz equations are immediate and straightforward. This was done already in [23]. More recently, spectrum in the Penrose limit [54], various SU(2|2) closed subsectors [32], all loop Bethe ansatz equations [55], and finite-size effects [56] were studied. With these developments, it would be interesting to explore precision tests for the new correspondence proposed by ABJM.

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A. Notation, convention and Feynman rules

A.1 Notation and convention

- $\mathbb{R}^{1,2}$ metric:

$$
\begin{aligned}
g_{mn} &= \mathrm{diag}(-,+,+) \qquad\qquad && \text{with} \quad m,n = 0,1,2. \\
\epsilon^{012} &= -\epsilon_{012} = +1 && \qquad\qquad\qquad\qquad\qquad\qquad \text{(A.1)} \\
\epsilon^{mpq}\epsilon_{mrs} &= -(\delta^p_r\delta^q_s - \delta^p_s\delta^q_r); && \epsilon^{mpq}\epsilon_{mpr} = -2\delta^q_r
\end{aligned}
$$

- $\mathbb{R}^{1,2}$ Majorana spinor and Dirac matrices:

$$\begin{aligned}
\psi &\equiv \text{two-component Majorana spinor} \\
\psi^\alpha &= \epsilon^{\alpha\beta}\psi_\beta, \qquad \psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta \qquad \text{where} \quad \epsilon^{\alpha\beta} = -\epsilon_{\alpha\beta} = i\sigma^2 \\
\gamma_\alpha^{m\beta} &= (i\sigma^2, \sigma^3, \sigma^1), \quad (\gamma^m)_{\alpha\beta} = (-\mathbb{I}, \sigma^1, -\sigma^3) \quad \text{obeying} \quad \gamma^m\gamma^n = g^{mn} - \epsilon^{mnp}\gamma_p.
\end{aligned} \tag{A.2}$$

A.2 ABJM theory

- Gauge and global symmetries:

$$\begin{aligned}
\text{gauge symmetry}: &\quad \mathrm{U}(N) \otimes \overline{\mathrm{U}(N)} \\
\text{global symmetry}: &\quad \mathrm{SU}(4)
\end{aligned} \tag{A.3}$$

We denote trace over $\mathrm{U}(N)$ and $\overline{\mathrm{U(N)}}$ as Tr and $\overline{\mathrm{Tr}}$, respectively.

- On-shell fields are gauge fields, complexified Hermitian scalars and Majorana spinors ($I = 1, 2, 3, 4$):

$$\begin{aligned}
A_m &: \text{ Adj } (\mathrm{U}(N)); \qquad \overline{A}_m : \quad \text{Adj } \overline{\mathrm{U}(N)} \\
Y^I &= (X^1 + iX^5, X^2 + iX^6, X^3 - iX^7, X^4 - iX^8): \qquad (\mathbf{N}, \overline{\mathbf{N}}; \mathbf{4}) \\
Y_I^\dagger &= (X^1 - iX^5, X^2 - iX^6, X^3 + iX^7, X^4 + iX^8): \qquad (\overline{\mathbf{N}}, \mathbf{N}; \overline{\mathbf{4}}) \\
\Psi_I &= (\psi^2 + i\chi^2, -\psi^1 - i\chi^1, \psi_4 - i\chi_4, -\psi_3 + i\chi_3): \qquad (\mathbf{N}, \overline{\mathbf{N}}; \overline{\mathbf{4}}) \\
\Psi^{\dagger I} &= (\psi_2 - i\chi_2, -\psi_1 + i\chi^1, \psi^4 + i\chi^4, -\psi^3 - i\chi^3): \qquad (\overline{\mathbf{N}}, \mathbf{N}; \mathbf{4})
\end{aligned} \tag{A.4}$$

- action:

$$\begin{aligned}
I = \int_{\mathbb{R}^{1,2}} \Big[&\frac{k}{4\pi}\epsilon^{mnp}\mathrm{Tr}\Big(A_m\partial_n A_p + \frac{2i}{3}A_m A_n A_p\Big) - \frac{k}{4\pi}\epsilon^{mnp}\overline{\mathrm{Tr}}\Big(\overline{A}_m\partial_n\overline{A}_p + \frac{2i}{3}\overline{A}_m\overline{A}_n\overline{A}_p\Big) \\
&+ \frac{1}{2}\overline{\mathrm{Tr}}\Big(-(D_mY)_I^\dagger D^m Y^I + i\Psi^{\dagger I}\not{D}\Psi_I\Big) + \frac{1}{2}\mathrm{Tr}\Big(-D_mY^I(D^mY)_I^\dagger + i\Psi_I\not{D}\Psi^{\dagger I}\Big) \\
&- V_{\mathrm{F}} - V_{\mathrm{B}} \Big]
\end{aligned} \tag{A.5}$$

Here, covariant derivatives are defined as

$$D_mY^I = \partial_mY^I + iA_mY^I - iY^I\overline{A}_m\,, \quad D_mY_I^\dagger = \partial_mY_I^\dagger + i\overline{A}_mY_I^\dagger - iY_I^\dagger A_m \tag{A.6}$$

and similarly for fermions $\Psi_I, \Psi^{\dagger I}$. Potential terms are

$$\begin{aligned}
V_{\mathrm{F}} = &\frac{2\pi i}{k}\overline{\mathrm{Tr}}\Big[Y_I^\dagger Y^I\Psi^{\dagger J}\Psi_J - 2Y_I^\dagger Y^J\Psi^{\dagger I}\Psi_J + \epsilon^{IJKL}Y_I^\dagger\Psi_JY_K^\dagger\Psi_L\Big] \\
&- \frac{2\pi i}{k}\mathrm{Tr}\Big[Y^IY_I^\dagger\Psi_J\Psi^{\dagger J} - 2Y^IY_J^\dagger\Psi_I\Psi^{\dagger J} + \epsilon_{IJKL}Y^I\Psi^{\dagger J}Y^K\Psi^{\dagger L}\Big]
\end{aligned} \tag{A.7}$$

and

$$V_{\rm B} = -\frac{1}{3}\left(\frac{2\pi}{k}\right)^2 \overline{\rm Tr}\Big[Y_I^\dagger Y^J Y_J^\dagger Y^K Y_K^\dagger Y^I + Y_I^\dagger Y^I Y_J^\dagger Y^J Y_K^\dagger Y^K \\ +4 Y_I^\dagger Y^J Y_K^\dagger Y^I Y_J^\dagger Y^K - 6 Y_I^\dagger Y^I Y_J^\dagger Y^K Y_K^\dagger Y^J \Big] \qquad \text{(A.8)}$$

At quantum level, since the Chern-Simons term shifts by integer multiple of $8\pi^2$, not only N but also k should be integrally quantized. To suppress the cluttering 2π factors, we also use the notation $\kappa = \frac{k}{2\pi}$. At large N, we expand the theory and physical observables in double series of

$$g_{\rm st} = \frac{1}{N}, \qquad \lambda = \frac{N}{k} = \frac{N}{2\pi\kappa} \qquad \text{(A.9)}$$

by treating them as continuous perturbation parameters.

A.3 Feynman rules

- We adopt Lorentzian Feynman rules and manipulate all Dirac matrices and ϵ_{mnp} tensor expressions to scalar integrals. For actual evaluation of these integrals, we shall go the Euclidean space integral by the Wick rotation, which corresponds to $x^0 \to -i\tau$. In the momentum space, this means we change the contour of p_0 to the imaginary axis following the standard Wick rotation. Then in terms of integration measure, we simply replace $\mathrm{d}^{2\omega}k \to i\mathrm{d}^{2\omega}k_{\rm E}$ together with $p^2 \to +p_{\rm E}^2$. The procedure is known to obey Slavnov-Taylor identity, at least to two loop order.

- We choose covariant gauge fixing condition for both gauge groups:

$$\partial^m A_m = 0 \qquad \text{and} \qquad \partial^m \overline{A}_m = 0 \qquad \text{(A.10)}$$

and work in Feynman gauge by setting the gauge parameter ξ to unity. Accordingly, we introduce a pair of Faddeev-Popov ghosts $c, \overline{c}$ and their conjugates, and add to I the ghosts action:

$$I_{\rm ghost} = \int_{\mathbb{R}^{2,1}} \Big[{\rm Tr}\partial^m c^* D_m c + \overline{\rm Tr}\partial^m \overline{c}^* D_m \overline{c} \Big] \qquad \text{(A.11)}$$

Here, $D_m c = \partial_m c + i[A_m, c]$ and $D_m\overline{c} = \partial_m \overline{c} + i[\overline{A}_m, \overline{c}]$.

- Propagators in $\mathrm{U}(N) \times \overline{\mathrm{U}(N)}$ matrix notation:

$$\begin{aligned}
&\text{gauge propagator}: && \Delta_{mn}(p) = \frac{2\pi}{k}\mathbb{I}\,\frac{\epsilon_{mnr}p^r}{p^2 - i\epsilon} \\
&\text{scalar propagator}: && D_I{}^J(p) = \delta_I^J \frac{-i}{p^2 - i\epsilon} \\
&\text{fermion propagator}: && S^I{}_J(p) = \delta_J^I \frac{i\not{p}}{p^2 - i\epsilon} \\
&\text{ghost propagator}: && K(p) = \frac{-i}{p^2 - i\epsilon}
\end{aligned} \qquad \text{(A.12)}$$

- Interaction vertices are obtained by multiplying $i = \sqrt{-1}$ to nonlinear terms of the Lagrangian density. Note that the paramagnetic coupling of gauge fields to scalar fields has the invariance property under simultaneous exchange between A_m, Y^I and $\overline{A}_m, Y_I^\dagger$.

B. Two-loop computations

B.1 Two-loop integrals

We first tabulate various Feynman integrals that appear recurrently among two-loop diagrams. They are all evaluated straightforwardly by Feynman parametrization

$$\frac{1}{A^a B^b} = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int\int \mathrm{d}x\mathrm{d}y\delta(1-x-y)\frac{x^{a-1}y^{b-1}}{(Ax+By)^{a+b}}. \tag{B.1}$$

We use dimensional regularization by shifting the spacetime dimension to $d = 2\omega = 3 - \epsilon$. The ultraviolet divergence shows up as a simple pole $1/\epsilon$. It is related to the momentum space cutoff Λ as

$$\frac{1}{\epsilon} := 2\log\Lambda\,. \tag{B.2}$$

In the following, we collect factors arising from propagators in parenthesis and those from vertices in square bracket. We have the following integrals:

- $$\begin{aligned} I_1 &= \int \frac{\mathrm{d}^{2\omega}k}{(2\pi)^{2\omega}}\frac{\mathrm{d}^{2\omega}\ell}{(2\pi)^{2\omega}}\frac{1}{(k+\ell)^2}\frac{1}{k^2}\frac{1}{\ell^2} \\ &= \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^{2\omega}\ell}{(2\pi)^{2\omega}}\frac{1}{\ell^2}\int\frac{\mathrm{d}^{2\omega}k}{(2\pi)^{2\omega}}\frac{1}{[k^2+2xk\cdot\ell+x\ell^2]^2} \\ &= -\frac{1}{8\pi}\int_0^1 \mathrm{d}x\frac{1}{\sqrt{x(1-x)}}\int\frac{\mathrm{d}^{2\omega}\ell}{(2\pi)^{2\omega}}\frac{1}{\sqrt{(k^2)^3}} \\ &= +\frac{1}{8}\frac{1}{4\pi^2}\frac{1}{\epsilon}. \end{aligned} \tag{B.3}$$

The integral that appears in fermion and gauge boson exchange diagrams is:

- $$I_2 = \int \frac{\mathrm{d}^{2\omega}k}{(2\pi)^{2\omega}}\frac{\mathrm{d}^{2\omega}\ell}{(2\pi)^{2\omega}}\frac{1}{(k^2)^2}\frac{2(k+\ell)\cdot\ell}{(k+\ell)^2\ \ell^2}\,. \tag{B.4}$$

We perform the ℓ integral first after using the Feynman reparametrization:

$$\begin{aligned} I_2 &= \int_0^1 \mathrm{d}x\int\frac{\mathrm{d}^{2\omega}k}{(2\pi)^{2\omega}}\frac{1}{(k^2)^2}\int\frac{\mathrm{d}^{2\omega}\ell}{(2\pi)^{2\omega}}\frac{2\ell\cdot(\ell+k)}{[\ell^2+2xk\cdot\ell+xk^2]^2} \\ &= -\frac{1}{8\pi}\int_0^1\mathrm{d}x\frac{\sqrt{x}}{\sqrt{1-x}}\int\frac{\mathrm{d}^{2\omega}k}{(2\pi)^{2\omega}}\frac{1}{k^3} \\ &= -\frac{1}{8}\frac{1}{4\pi^2}\frac{1}{\epsilon}\,, \end{aligned} \tag{B.5}$$

where for the second equality, we have used the integral,

$$\int \frac{\mathrm{d}^{2\omega}\ell}{(2\pi)^{2\omega}} \frac{\ell_m \ell_n}{[\ell^2 + 2x\ell\cdot k + k^2]^2} = \frac{1}{(4\pi)^{3/2}}\left[\frac{x^2 k_m k_n \Gamma(1/2)}{[x(1-x)k^2]^{1/2}} + \frac{g_{mn}}{2}\frac{\Gamma(-1/2)}{[x(1-x)k^2]^{-1/2}}\right]$$
$$\int \frac{\mathrm{d}^{2\omega}\ell}{(2\pi)^{2\omega}} \frac{\ell_m}{[\ell^2 + 2x\ell\cdot k + k^2]^2} = -\frac{1}{(4\pi)^{3/2}}\frac{x k_m \Gamma(1/2)}{[x(1-x)k^2]^{1/2}}\,. \tag{B.6}$$

If one exchanges the order of integrations, there may appear an infrared singularity. However by introducing infrared regulator mass m, one may get the same result in the limit $\omega \to 3/2$ and $m \to 0$.

In the gauge boson exchange diagram, the following integral appears:

- $$I_3 = \int \frac{\mathrm{d}^{2\omega}k}{(2\pi)^{2\omega}} \int \frac{\mathrm{d}^{2\omega}\ell}{(2\pi)^{2\omega}} \frac{1}{(k+\ell)^2}\frac{1}{(k^2)^2}\frac{1}{(\ell^2)^2}[(k\cdot\ell)^2 - k^2\ell^2] \equiv I_{3,A} - I_{3,B}$$

We evaluated them as follows:

$$\begin{aligned}
I_{3,A} &= \int \frac{\mathrm{d}^{2\omega}k}{(2\pi)^{2\omega}}\frac{k_m k_n}{(k^2)^2}\int_0^1 \mathrm{d}x \frac{\Gamma(3)}{\Gamma(2)}\int \frac{\mathrm{d}^{2\omega}\ell}{(2\pi)^{2\omega}}\frac{(1-x)\ell_m\ell_n}{[\ell^2+2x\ell\cdot k+k^2]^3}\\
&= \int \frac{\mathrm{d}^{2\omega}k}{(2\pi)^{2\omega}}\frac{k_m k_n}{(k^2)^2}\int_0^1 \mathrm{d}x(1-x)\frac{1}{(4\pi)^{3/2}}\left[x^2 k_m k_n \frac{\Gamma(3/2)}{[x(1-x)k^2]^{3/2}} + \frac{g_{mn}}{2}\frac{\Gamma(1/2)}{[x(1-x)k^2]^{1/2}}\right]\\
&= \frac{1}{16}\frac{1}{4\pi^2}\frac{1}{\epsilon}.\\
I_{3,B} &= \int \frac{\mathrm{d}^{2\omega}}{(2\pi)^{2\omega}}\frac{1}{k^2}\int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^{2\omega}\ell}{(2\pi)^{2\omega}}\frac{1}{[\ell^2+2x\ell\cdot k+xk^2]^2}\\
&= \frac{\Gamma(1/2)}{(4\pi)^{3/2}\Gamma(2)}\int_0^1 \mathrm{d}x \frac{1}{x(1-x)}\int \frac{\mathrm{d}^{2\omega}k}{(2\pi)^{2\omega}}\frac{1}{(k^2)^{3/2}}\\
&= \frac{1}{8}\frac{1}{4\pi^2}\frac{1}{\epsilon}
\end{aligned} \tag{B.7}$$

Hence,

$$I_3 = \left(\frac{1}{16} - \frac{1}{8}\right)\frac{1}{4\pi^2}\frac{1}{\epsilon} = -\frac{1}{16}\frac{1}{4\pi^2}\frac{1}{\epsilon}. \tag{B.8}$$

B.2 Contribution from sextet scalar potential

The Lagrangian contains sextet scalar interaction $-V_{\text{scalar}}$. Three of the scalar fields couple to $\mathcal{O}$ and the rest three to $\mathcal{O}^\dagger$. With $\mathrm{U}(N)$ and $\overline{\mathrm{U(N)}}$ index loops, combinatorial factors are given by

$$-3\cdot N^2\Big[2\mathbb{I}^{\otimes^3} - 4\mathbb{P}_{13}\otimes\mathbb{I}_2 - \mathbb{K}_{12}\otimes\mathbb{I}_3 - \mathbb{I}\otimes\mathbb{K}_{23}\otimes\mathbb{I}_2 + 2\mathbb{K}_{13}\otimes\mathbb{K}_{12} + 2\mathbb{K}_{12}\otimes\mathbb{K}_{13}\Big] \tag{B.9}$$

There are three scalar propagators and one interaction vertex, contributing factors

$$\frac{1}{3\kappa^2}(i)^3\ [i]\cdot N^2 = \frac{4\pi^2}{3}\lambda^2 \tag{B.10}$$

The remaining 2-loop integral is given by I_1. Summing over all contributions, the scalar sextet interaction gives rise to 2-loop dilatation operator

$$H_{\mathrm{B}} = \frac{\lambda^2}{2}\sum_{\ell=1}^{2L}\Big[\mathbb{I} - 2\mathbb{P}_{\ell,\ell+2} - \mathbb{K}_{\ell,\ell+1} + \mathbb{P}_{\ell,\ell+2}\mathbb{K}_{\ell,\ell+1} + \mathbb{K}_{\ell,\ell+1}\mathbb{P}_{\ell,\ell+1}\Big] \tag{B.11}$$

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B.3 Contribution from two-site interactions

In this appendix we shall present the full detailed computation of the two site interactions. First let us compute the Yukawa two-site interactions. The nonvanishing Yukawa interaction leads to only a $\mathbb{K}$-type interaction. The relevant Feynman diagram is depicted in figure 2b.

With two Yukawa interaction components and one $\mathrm{U}(N)$ and one $\overline{\mathrm{U(N)}}$ color traces, combinatorial factors are gathered as

$$\frac{1}{2!} \cdot 2 \cdot N^2 = N^2. \tag{B.12}$$

There are four propagators and two vertices. This yields numerical factors

$$(-i)^2(i)^2[i]^2\left(\pm\frac{2i}{\kappa}\right)^2 \cdot (-)_{\mathrm{FD}} \tag{B.13}$$

where the subscript $(\ )_{\mathrm{FD}}$ signifies the Fermi-Dirac statistics minus sign. The loop integral is given by

$$i^2 \int \frac{\mathrm{d}^{2\omega}k}{(2\pi)^{2\omega}} \frac{\mathrm{d}^{2\omega}\ell}{(2\pi)^{2\omega}} \frac{1}{(-(k)^2)^2} \mathrm{tr}\left(\frac{\not{\ell}}{\ell^2}\frac{\not{k}+\not{\ell}}{(k+\ell)^2}\right) \tag{B.14}$$

where the i^2 factor comes from the analytic continuation of the integration measure.

After taking the gamma matrix trace $\mathrm{tr}\gamma^m\gamma^n = 2g^{mn}$, this integral equals to $-I_2$ in (B.5).

Hence putting everything together, one has

$$\lambda^2\frac{(-1)}{2\epsilon}\mathbb{K} \tag{B.15}$$

for the Yukawa two-site interactions. The contribution to the operator renormalization is negative of this: Therefore, the Yukawa contribution is

$$H_{\mathrm{F}} = \lambda^2 \sum_{\ell=1}^{2L} \mathbb{K}_{\ell,\ell+1} \tag{B.16}$$

We now evaluate the gauge two-site interactions.

The gauge boson interactions contribute both $\mathbb{K}$ and $\mathbb{I}$ type diagrams to the dilatation operator. Let us begin with $\mathbb{K}$ type contribution. The relevant diagram is in figure 2c. It has combinatorial factors

$$\frac{1}{2!} \cdot 2 \cdot N^2 = N^2. \tag{B.17}$$

There are three boson propagators, two gauge propagators and one seagull interaction vertex. So, numerical factors are given by

$$(-i)^3 \cdot [-i]^3 \cdot (\pm 1)^2\left(\frac{1}{\kappa}\right)^2 = -\frac{4\pi^2}{k^2} \tag{B.18}$$

where the last factor accounts for the ($\pm$) relative sign of $\mathrm{U}(N)$ and $\overline{\mathrm{U(N)}}$ Chern-Simons term. It is important to note that the gauge field propagator in momentum space has no $i = \sqrt{-1}$. The loop integral reads

$$i^2 \int \frac{\mathrm{d}^{2\omega}k}{(2\pi)^{2\omega}} \frac{\mathrm{d}^{2\omega}\ell}{(2\pi)^{2\omega}} \frac{1}{(k+\ell)^2} \frac{1}{(k^2)^2} \frac{1}{(\ell^2)^2} (\epsilon_{mnp}(k+2\ell)^n k^p) g^{mq} (\epsilon_{qrs}(k+2\ell)^r (-k)^s) \quad \text{(B.19)}$$

where again the i^2 factor comes from the Euclidean rotation. Using the identity $g^{mq}\epsilon_{mnp}\epsilon_{qrs} = -(g_{nr}g_{ps} - g_{ns}g_{pr})$, we find that the integral is the same as $4I_3$.

Hence putting everything together, one has

$$-\frac{\lambda^2}{2}\frac{(-1)}{2\epsilon}\mathbb{K} \quad \text{(B.20)}$$

for the gauge two site $\mathbb{K}$ contributions and, for the operator renormalization,

$$-\frac{\lambda^2}{2}\frac{1}{2\epsilon}\mathbb{K}\,. \quad \text{(B.21)}$$

There are also contributions to $\mathbb{I}$ from t-channel exchange of diamagnetic gauge boson interaction. The corresponding Feynman diagram is depicted in Fig 2a. There are two scalar propagators, two gauge boson propagators and two diamagnetic vertices. Note again, for Chern-Simons theory, gauge boson propagator has no i in momentum space. So, the combinatorial factor is

$$\frac{1}{2!}2 \cdot (-i)^2 \cdot [i]^2 \cdot N^2 \cdot \left(\frac{1}{\kappa}\right)^2 = (4\pi^2)\lambda^2. \quad \text{(B.22)}$$

The loop integral reads

$$i^2 \int \frac{\mathrm{d}^{2\omega}k}{(2\pi)^{2\omega}} \frac{\mathrm{d}^{2\omega}\ell}{(2\pi)^{2\omega}} \frac{1}{(k^2)^2} \frac{\epsilon^{mna}(k+\ell)_a}{(k+\ell)^2} \frac{\epsilon_{mn}{}^b \ell_b}{\ell^2} \quad \text{(B.23)}$$

Using the identity $\epsilon^{mna}\epsilon_{mn}{}^b = -2g^{ab}$, we find that this integral is the same as $-I_2$. There are identical contributions from each letter (with alternating $\mathrm{U}(N)$ and $\overline{\mathrm{U(N)}}$ gauge boson exchanges), we find the contribution as

$$-\frac{\lambda^2}{4}\frac{(-1)}{2\epsilon}\mathbb{I}\,. \quad \text{(B.24)}$$

The corresponding operator renormalozation contribution is

$$-\frac{\lambda^2}{4}\frac{1}{2\epsilon}\mathbb{I}\,. \quad \text{(B.25)}$$

Hence there are two gauge two-site contributions. Using $1/(2\epsilon) = \ln\Lambda$, the gauge two-site contributions to the anomalous dimension are summarized as

$$H_{\text{gauge}} = \sum_{\ell=1}^{2L}\left[-\frac{1}{4}\mathbb{I} - \frac{1}{2}\mathbb{K}_{\ell,\ell+1}\right]\lambda^2\,. \quad \text{(B.26)}$$

B.4 Contributions of wave function renormalization

The first one involves diamagnetic gauge interactions. The relevant Feynman diagrams are in figure 3. As scalar fields are bi-fundamentals, there are processes involving $\mathrm{U}(N)$ gauge boson pair, $\overline{\mathrm{U(N)}}$ gauge boson pair, and one $\mathrm{U}(N)$ gauge boson and one $\overline{\mathrm{U}(N)}$ gauge boson pair, which are respectively corresponding to figure 3a, figure 3b and figure 3c. Taking account of opposite relative sign between gauge boson propagators for $\mathrm{U}(N)$ and $\overline{\mathrm{U(N)}}$ and of different combinatorial weight of diamagnetic coupling terms, the numerical factor reads

$$\frac{1}{2!}2\cdot(-i)\cdot[i]^2\cdot[(-)^2\cdot(+)^2+(+)^2\cdot(-)^2+(+)(-)\cdot(-2)^2]N^2\left(\frac{1}{\kappa}\right)^2=-2i(4\pi^2)\lambda^2 \quad \text{(B.27)}$$

Denote momentum of the external scalar field as p^m. Then, loop integral reads

$$i^2\int\frac{\mathrm{d}^{2\omega}k}{(2\pi)^{2\omega}}\frac{\mathrm{d}^{2\omega}\ell}{(2\pi)^{2\omega}}\frac{1}{(k+\ell+p)^2}\frac{\epsilon^{mna}k_a}{k^2}\frac{\epsilon_{mn}{}^b\ell_b}{\ell^2}\,. \quad \text{(B.28)}$$

For the evaluation of this integral, let us introduce

$$\begin{aligned} I_G(p) &= \int\frac{\mathrm{d}^{2\omega}k}{(2\pi)^{2\omega}}\frac{\mathrm{d}^{2\omega}\ell}{(2\pi)^{2\omega}}\frac{1}{(k+\ell+p)^2}\frac{2k\cdot\ell}{k^2\;\ell^2}\\ &=2\int\frac{\mathrm{d}^{2\omega}\ell}{(2\pi)^{2\omega}}\frac{1}{\ell^2}\int_0^1\mathrm{d}x\int\frac{\mathrm{d}^{2\omega}k}{(2\pi)^{2\omega}}\frac{k\cdot\ell}{[(k+x(\ell+p))^2+x(1-x)(\ell+p)^2]^2}\\ &=-\frac{1}{4\pi}\int_0^1\mathrm{d}x\sqrt{\frac{x}{1-x}}\int\frac{\mathrm{d}^{2\omega}\ell}{(2\pi)^{2\omega}}\frac{\ell\cdot(\ell+p)}{\sqrt{(\ell+p)^2}}. \end{aligned} \quad \text{(B.29)}$$

The x-integral is finite and equals to $\pi/2$. The remaining ℓ-integral can be performed by applying Feynman's parametrization. In dimensional regularization, we have

$$\begin{aligned} -\frac{1}{8}\frac{\Gamma(3/2)}{\Gamma(1/2)}\int_0^1\mathrm{d}y\frac{1}{\sqrt{y}}\int\frac{\mathrm{d}^{2\omega}\ell}{(2\pi)^{2\omega}}\frac{\ell\cdot p}{[\ell^2+2x\ell\cdot p+xp^2]^{3/2}}&\\ =-\frac{1}{16}\int_0^1\frac{\mathrm{d}y}{\sqrt{y}}\left[-\frac{\Gamma(\epsilon)}{(4\pi)^\omega\Gamma(3/2)}\frac{yp^2}{(y(1-y)p^2)^\epsilon}\right]& \end{aligned} \quad \text{(B.30)}$$

Taking $\epsilon=3/2-\omega\to 0$, this integral equals to

$$I_G=\frac{1}{24}\frac{1}{4\pi^2}\frac{1}{\epsilon}. \quad \text{(B.31)}$$

Putting together, we thus find that these diagrams contribute to the wave function renormalization as

$$-\frac{1}{12}\lambda^2\frac{1}{\epsilon}(ip^2) \quad \text{(B.32)}$$

Consider next two diagrams involving four paramagnetic couplings. Planar diagrams involve two vertices from $\mathrm{U}(N)$ and two from $\overline{\mathrm{U(N)}}$, as shown in figure 4. Taking care of opposite relative sign of gauge boson propagators between $\mathrm{U}(N)$ and $\overline{\mathrm{U}(N)}$ and that there are three internal scalar propagators, we have combinatorial factors

$$\frac{1}{(2!)^2}2^2\cdot(+)(-)\cdot(-i)^3[i]^2\left(\frac{1}{\kappa}\right)^2N^2(2)=-2i(4\pi^2)\lambda^2. \quad \text{(B.33)}$$

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where we put an additional factor two because there are two such diagrams. With external momentum p^m, the loop integral read

$$i^2 \int \frac{\mathrm{d}^{2\omega}k}{(2\pi)^{2\omega}} \frac{\mathrm{d}^{2\omega}\ell}{(2\pi)^{2\omega}} \frac{\epsilon^{mnq}(\ell+2p)_m(p+2k+2\ell)_n \ell_q \epsilon^{abc}(2\ell+k+2p)_a(k+2p)_b k_c}{(k+\ell+p)^2(\ell+p)^2(k+p)^2 k^2 \ell^2} . \qquad \text{(B.34)}$$

This integral can be integrated without further assumption but we note that the numerator of the integrand is already quadratic in p_m. Using the isotropy of the system, we replace

$$p_a p_b \quad \rightarrow \quad \frac{p^2}{3} g_{ab} \qquad \text{(B.35)}$$

and then set p to zero in the remaining integral. One may show that the results from the both methods agree precisely with each other.

Thus the integral becomes

$$-\frac{16}{3}p^2 \;\; I_3 = \frac{p^2}{12\pi^2}\frac{1}{\epsilon} . \qquad \text{(B.36)}$$

Putting all the factor together, one has

$$-\frac{2}{3}\lambda^2 \frac{1}{\epsilon}(ip^2) \qquad \text{(B.37)}$$

There are two diagrams involving Chern-Simons cubic coupling. The contributions of $\mathrm{U}(N)$ and $\overline{\mathrm{U(N)}}$ are added up with an equal weight. The Feynman diagrams are in figure 5.

The relevant combinatorics is

$$\frac{3! \cdot 3}{3!}(-i)^2[i]^4 N^2 \left(\frac{(\pm)\kappa i}{3}\right)\left(\frac{(\pm 1)}{\kappa}\right)^3 (2) = -2i(4\pi^2)\lambda^2 \qquad \text{(B.38)}$$

where the last factor two takes care of the $\overline{\mathrm{U(N)}}$ contribution. The loop integral becomes

$$i^2 \int \frac{\mathrm{d}^{2\omega}k}{(2\pi)^{2\omega}} \frac{\mathrm{d}^{2\omega}\ell}{(2\pi)^{2\omega}} \frac{\epsilon^{mqn}\epsilon_{mar}(2p+k)^a k^r \epsilon_{nbs}(2p-\ell+k)^b(-k-\ell)^s \epsilon_{qct}(2p-\ell)^b \ell^t}{(p-\ell)^2(k+\ell)^2(k+p)^2 k^2 \ell^2} . \qquad \text{(B.39)}$$

Using the rule of

$$p_a p_b \rightarrow \frac{\delta_{ab}}{3} p^2 ,$$

the integral becomes

$$i^2 \frac{8p^2}{3} \int \frac{\mathrm{d}^{2\omega}k}{(2\pi)^{2\omega}} \frac{\mathrm{d}^{2\omega}\ell}{(2\pi)^{2\omega}} \frac{-k^2\ell^2 + (k\cdot\ell)^2}{(k+\ell)^2(k^2)^2(\ell^2)^2} . \qquad \text{(B.40)}$$

Using I_3, one finds

$$\frac{p^2}{6}\frac{1}{4\pi^2}\frac{1}{\epsilon} . \qquad \text{(B.41)}$$

Therefore, the whole contribution combining the combinatorics becomes

$$-\frac{1}{3}\lambda^2 \frac{1}{\epsilon} ip^2 . \qquad \text{(B.42)}$$

There are also diagrams involving paramagnetic and diamagnetic couplings. Their net combinatorial factor is nonzero, but the loop integral vanishes identically.

Let us now turn to the Yukawa contributions. First consider the Feynman diagrams in figure 6a and figure 6b. Within the planar diagram, both fermion can be joined either $\mathrm{U}(N)$ side (figure 6a) and $\overline{\mathrm{U(N)}}$ side (figure 6b). The joining using the first two terms has a factor 4 from the SU(4) index contraction. Then the cross terms between the first two and the second two terms in total have a factor -4. Hence one can check that this cross contributions cancel precisely the those from the first two.

By combining the second two of the Yukawa potential, for the $\mathrm{U}(N)$ and $\overline{\mathrm{U(N)}}$ side, we have combinational factors

$$\frac{1}{2!}2 \cdot (-i \cdot i^2)[i]^2 \cdot \left(\frac{2i}{\kappa}\right)^2 N^2 (-)_{\mathrm{FD}} \times 8 = -32i(4\pi^2)\lambda^2. \tag{B.43}$$

where the extra factor eight comes from one contraction of SU(4) index and the doubling by $\mathrm{U}(N)$ and $\overline{\mathrm{U(N)}}$.

Then the remaining integral has the expression,

$$i^2 \int \frac{\mathrm{d}^{2\omega}k}{(2\pi)^{2\omega}} \frac{\mathrm{d}^{2\omega}\ell}{(2\pi)^{2\omega}} \frac{1}{(p+k-\ell)^2} \frac{\mathrm{tr}\, \not{\ell}\, \not{k}}{k^2\ \ell^2} \tag{B.44}$$

which is same as I_G. Therefore the whole contribution is

$$-\frac{4}{3}\lambda^2 \frac{1}{\epsilon}(ip^2)\,. \tag{B.45}$$

For the wave function renormalization, the third two of Yukawa potential also contribute. The diagram is in figure 6c. It has combinatoric factors,

$$(2!)^2 \cdot (-i) \cdot (i)^2 [i]^2 \cdot \left(\frac{i}{\kappa}\right)\left(\frac{-i}{\kappa}\right) N^2 (-)_{\mathrm{FD}} \times (-6) = -24i(4\pi^2)\lambda^2\,, \tag{B.46}$$

where $(2!)^2$ is the usual symmetry factor of the Feynman diagram. The last factor (-6) comes from the following SU(4) index contraction

$$\epsilon_{IABC}\epsilon^{JCBA} = -6\ \delta_I^J \tag{B.47}$$

where I is for the incoming and the J for the outgoing scalar SU(4) indices.

Then the remaining integral takes precisely the same from:

$$i^2 \int \frac{\mathrm{d}^{2\omega}k}{(2\pi)^{2\omega}} \frac{\mathrm{d}^{2\omega}\ell}{(2\pi)^{2\omega}} \frac{1}{(p+k-\ell)^2} \frac{\mathrm{tr}\, \not{\ell}\, \not{k}}{k^2\ \ell^2} \tag{B.48}$$

which is again the same as I_G. Therefore the whole contribution is

$$-\lambda^2 \frac{1}{\epsilon}(ip^2)\,. \tag{B.49}$$

Finally, there are the vacuum polarization contributions of the gauge loop. The relevant diagrans are depicted in figure 7.

As we shall explain in the following appendix, the self energy correction for both A and $\overline{A}$ gauge fields is given by

$$i\,\Pi_{ab}(k) = 8i\left[\frac{k_a k_b - g_{ab}k^2}{16k}\right], \tag{B.50}$$

where the factor eight comes from the four complex scalars and fermions with an equal weight.

For the relevant diagram of figure 7, the combinatorics factor reads

$$\frac{2!}{2!}\cdot(-i)[i]^2\cdot\left(\frac{1}{\kappa}\right)^2 N^2\times(2) = 2i(4\pi^2)\lambda^2\,, \tag{B.51}$$

where the last factor two comes from the doubling by replacing A gauge by the $\overline{A}$ gauge field. The remaining Feynman integrals takes the from,

$$\begin{aligned} i\int\frac{\mathrm{d}^{2\omega}k}{(2\pi)^{2\omega}}\frac{\epsilon^{amn}(2p+k)_m(-k)_n\epsilon^{bij}(k+2p)_i k_j\, i\,\Pi_{ab}(k)}{(k+p)^2(k^2)^2}\,, \\ = 2\int\frac{\mathrm{d}^{2\omega}k}{(2\pi)^{2\omega}}\frac{k^2p^2-(k\cdot p)^2}{(k+p)^2\ k^3} \end{aligned} \tag{B.52}$$

where we have a single i produced by the Euclidean rotation.

By the dimensional regularization, this leads to

$$p^2\frac{1}{3\pi^2}\frac{1}{\epsilon}\,. \tag{B.53}$$

Hence the total contribution reads

$$\frac{8}{3}\lambda^2\frac{1}{\epsilon}(ip^2)\,. \tag{B.54}$$

All the remaining diagrams, one may prove that their contribution is identically zero after the dimensional regularization.

Finally we add up all the above contributions to the wave function renormalization and find that

$$-\frac{3}{4}\lambda^2\frac{1}{\epsilon}(-)(-ip^2)\,. \tag{B.55}$$

Since the counter term is a negative of this, the two-loop scalar wave function renormalization becomes

$$Z_s = 1-\frac{3}{4}\lambda^2\frac{1}{\epsilon} = 1-\frac{3}{4}\lambda^2(2\ln\Lambda)\,. \tag{B.56}$$

In order to get the operator renormalization factor, one has to take $Z_s^{\frac{1}{2}}$ out for each site, which corresponds to adding $-\frac{1}{2}$ of (B.56) to the interaction part of renormalization. The final contribution to the anomalous dimension is

$$H_Z = \lambda^2\sum_{\ell=1}^{2L}\left[\left(\frac{1}{12}+\frac{2}{3}+\frac{1}{3}\right)+\left(\frac{4}{3}+1\right)-\frac{8}{3}\right]\mathbb{I} = \lambda^2\sum_{\ell=1}^{2L}\frac{3}{4}\mathbb{I}\,. \tag{B.57}$$

For the gauge two-loop contributions 1/12, 2/3, 1/3 including the gauge self-energy correction contribution 8/3, the two-loop Feynman diagram computation is carried out in ref. [57] for the U(1) case. One can check the precise agreement after taking care of the planarity factor and the number of matter degrees. Furthermore, ref. [58] deals with the two-loop Yukawa contribution to the scalar wave function renormalization for again U(1). This result is again matching with ours if one takes care of the planarity and the number of fermions.

B.5 One loop self energy correction to the gauge field

The self-energy correction enters in the same form for the $\mathrm{U}(N)$ and the $\overline{\mathrm{U(N)}}$ gauge fields. Therefore we focus on the correction to A gauge field only. At the one-loop level, the boson, the fermion, the gauge and the ghost loops may in general contribute to the gauge self-energy correction. In this appendix, we identify these self-energy contributions.

We begin with the scalar loop contribution. It is the sub-diagram of figure 7a. The momentum k plays the role of the external momentum. The self energy contribution reads

$$i\,\Pi^{s}_{ab}(k) = (i)^2[i\,]^2(4)i\int \frac{\mathrm{d}^{2\omega}\ell}{(2\pi)^{2\omega}}\frac{(2\ell+k)_a(2\ell+k)_b}{(k+\ell)^2\ \ell^2}\,, \tag{B.58}$$

where the extra factor 4 comes from the fact that 4 complex scalars are coupled to the gauge field. Using the dimensional regularization, one obtains

$$i\,\Pi^{s}_{ab}(k) = (4)i\left[\frac{k_ak_b-g_{ab}k^2}{16k}\right]. \tag{B.59}$$

Similarly, for the fermion loop, the self-energy contribution becomes

$$i\,\Pi^{f}_{ab}(k) = (i)^2[i\,]^2(4)(-)_{\mathrm{FD}}\,i\int \frac{\mathrm{d}^{2\omega}\ell}{(2\pi)^{2\omega}}\frac{\mathrm{Tr}\,\gamma_a\,(\not\ell+\not k)\,\gamma_b\not\ell}{(k+\ell)^2\ \ell^2}\,, \tag{B.60}$$

where again the extra factor four comes from the fact that there are 4 complex fundamental fermions. Using the γ matrix identity and the dimensional regularization, the contribution becomes

$$i\,\Pi^{f}_{ab}(k) = (4)i\left[\frac{k_ak_b-g_{ab}k^2}{16k}\right]. \tag{B.61}$$

Hence, each complex matter contributes by the same weight and sign.

One can continue the dimensions 2ω to four and obtain the vacuum polarization in four-dimensional Yang-Mills theories. The integration leads to the logarithmic divergence in this case contributing positively to the β-function of the Yang-Mills coupling. Again, boson and fermion contributions add up.

For the gluon self-energy contribution, we have

$$\begin{aligned} i\,\Pi^{A}_{ab}(k) &= (3)\cdot(3)[i^2]\left[\frac{i\kappa}{3}\right]^2\left[\frac{1}{\kappa}\right]^2(i)^2[i\,]^2(4)i\int\frac{\mathrm{d}^{2\omega}\ell}{(2\pi)^{2\omega}}\frac{\epsilon^{mbn}\epsilon^{jai}\epsilon_{imq}\epsilon_{njr}(\ell+k)^q\ell^r}{(k+\ell)^2\ \ell^2}\,,\\ &= i\int\frac{\mathrm{d}^{2\omega}\ell}{(2\pi)^{2\omega}}\frac{(\ell+k)_a\ell_b+(\ell+k)_b\ell_a}{(k+\ell)^2\ \ell^2}\,. \end{aligned} \tag{B.62}$$

It becomes

$$i\,\Pi^A_{ab}(k) = -i\left[\frac{k_a k_b + g_{ab}k^2}{32k}\right], \tag{B.63}$$

which alone does not respect the gauge invariance. However, there exists also the ghost loop contribution,

$$i\,\Pi^{\rm gh}_{ab}(k) = (i)^2[i\,]^2(-)i\int \frac{{\rm d}^{2\omega}\ell}{(2\pi)^{2\omega}}\frac{(\ell+k)_a\ell_b+(\ell+k)_b\ell_a}{(k+\ell)^2\;\ell^2}\,, \tag{B.64}$$

where we put the extra $(-)$ sign due to the ghost statistics. Therefore, the ghost contribution cancels out precisely the gauge loop contribution, reproducing the well-established result [43].

Again, analytically continuing to four dimensions, the integral expression for the gauge part changes while the ghost integral remains intact. With Yang-Mills couplings, both contributions no longer cancel each other but contribute negatively to the β-function.

C. Wrapping interactions for the two-sites

As in figure 8, there are occuring three kinds of wrapping interactions. First is the gauge interactions of two diamagnetic couplings in figure 8b. It is an $\mathbb{I}$ type interaction and happens, not for each site, but just once.

The combinatorial factor is

$$\frac{1}{2!}2\cdot(-i)^2\cdot[2i\,]^2\cdot(+)(-)\cdot N^2\cdot\left(\frac{1}{\kappa}\right)^2 = -(4)4\pi^2\lambda^2. \tag{C.1}$$

where (+)(-) accounts for the the relative $\mathrm{U}(N)$ and $\overline{\mathrm{U}(N)}$ Chern-Simons term and the factor two in the vertices takes care of the diamagnetic interaction.

The loop integral

$$i^2\int\frac{{\rm d}^{2\omega}k}{(2\pi)^{2\omega}}\frac{{\rm d}^{2\omega}\ell}{(2\pi)^{2\omega}}\frac{1}{(k^2)^2}\frac{\epsilon^{mna}(k+\ell)_a}{(k+\ell)^2}\frac{\epsilon_{mn}{}^b\ell_b}{\ell^2} \tag{C.2}$$

is the same as (B.23). So, the loop integral is evaluated as

$$\frac{1}{8}\frac{1}{4\pi^2}\frac{1}{\epsilon}\,. \tag{C.3}$$

Putting things together, we find the whole contribution as

$$\lambda^2\frac{(-1)}{2\epsilon}\mathbb{I}\,. \tag{C.4}$$

The corresponding operator renormalization contribution is

$$\lambda^2\frac{1}{2\epsilon}\mathbb{I}\,. \tag{C.5}$$

The second is for the K type gauge wrapping, whose Feynman diagram is depicted in figure 8c. It is doubling of the K-type interaction discussed for the general two-site gauge interactions. This doubling occurs due to the fact that, on the cylinder, one may have two different topology of the diagrams. Namely the diamagnetic interaction of the same gauge group may happen either one side or the other side, which is not possible for the infinite chains. From the previous result, the corresponding extra operator renormalization contribution is

$$-\lambda^2 \frac{1}{2\epsilon} \mathbb{K}\,, \tag{C.6}$$

where we take into account of the fact that the doubling occurs both for the $\mathrm{U}(N)$ and $\overline{\mathrm{U}(N)}$.

There is an additional wrapping interaction coming from the third two terms in the Yukawa potential. The Feynman diagram is in figure 8. In order to have proper contractions, one has to join operator site one (Y^{I_1}) to the site two (Y^{J_2}) whereas the operator site two $Y^\dagger_{I_2}$ to $Y^\dagger_{J_1}$. The corresponding ϵ tensors in the Yukawa interaction produce

$$\epsilon_{I_1 A J_2 B}\, \epsilon^{I_2 B J_1 A} = 2(\mathbb{I} - \mathbb{K})\,. \tag{C.7}$$

The combinatorial factors are gathered as

$$\frac{2!}{2!} \cdot 2 \cdot 2(-i)^2 (i)^2 [i\,]^2 \cdot N^2 \cdot \left(\frac{i}{\kappa}\right)\left(\frac{-i}{\kappa}\right) \cdot (-)_{\mathrm{FD}} = 4(4\pi^2) \cdot \lambda^2\,. \tag{C.8}$$

The loop integral is given by

$$i^2 \int \frac{\mathrm{d}^{2\omega}k}{(2\pi)^{2\omega}} \frac{\mathrm{d}^{2\omega}\ell}{(2\pi)^{2\omega}} \frac{1}{(-(k)^2)^2} \mathrm{tr}\left(\frac{\not{\ell}}{\ell^2} \frac{\not{k} + \not{\ell}}{(k+\ell)^2}\right), \tag{C.9}$$

which is the same as (B.14). By the loop integration, one gets

$$\frac{1}{8} \frac{1}{4\pi^2} \frac{1}{\epsilon}\,. \tag{C.10}$$

Hence putting everything together, one has

$$2\lambda^2 \frac{(-1)}{2\epsilon} (\mathbb{K} - \mathbb{I}) \tag{C.11}$$

for the Yukawa wrapping interactions. Therefore, the Yukawa contribution to the operator renormalization is

$$2\,\lambda^2 \frac{1}{2\epsilon} (\mathbb{K} - \mathbb{I})\,. \tag{C.12}$$

Adding up the gauge and Yukawa contributions, the wrapping interaction contribution to the two-site Hamiltonian is

$$H_{\mathrm{wrap}} = \mathbb{I} - \mathbb{K} + 2(\mathbb{K} - \mathbb{I}) = -\mathbb{I} + \mathbb{K}\,. \tag{C.13}$$

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